In the current chapter, you will study the motion of systems of particles.

The effective force of a particle is defined as the product of its mass and acceleration. It will be shown that the system of external forces acting on a system of particles is equipollent with the system of effective forces of the system.

The mass center of a system of particles will be defined and its motion described.

Application of the work-energy principle and the impulse-momentum principle to a system of particles will be described. Results obtained are also applicable to a system of rigidly connected particles, i.e., a rigid body.

Analysis methods will be presented for variable systems of particles, i.e., systems in which the particles included in the system change.
Application of Newton’s Laws. Effective Forces

• Newton’s second law for each particle $P_i$ in a system of $n$ particles,

\[
\vec{F}_i + \sum_{j=1}^{n} \vec{f}_{ij} = m_i \vec{a}_i
\]

\[
r_i \times \vec{F}_i + \sum_{j=1}^{n} (r_i \times \vec{f}_{ij}) = r_i \times m_i \vec{a}_i
\]

$\vec{F}_i = \text{external force}$  $\vec{f}_{ij} = \text{internal forces}$  $m_i \vec{a}_i = \text{effective force}$

• The system of external and internal forces on a particle is equivalent to the effective force of the particle.

• The system of external and internal forces acting on the entire system of particles is equivalent to the system of effective forces.

Vector Mechanics for Engineers: Dynamics

Application of Newton’s Laws. Effective Forces

• Summing over all the elements,

\[
\sum_{i=1}^{n} \vec{F}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \vec{f}_{ij} = \sum_{i=1}^{n} m_i \vec{a}_i
\]

\[
\sum_{i=1}^{n} (r_i \times \vec{F}_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} (r_i \times \vec{f}_{ij}) = \sum_{i=1}^{n} (r_i \times m_i \vec{a}_i)
\]

• Since the internal forces occur in equal and opposite collinear pairs, the resultant force and couple due to the internal forces are zero,

\[
\sum \vec{F}_i = \sum m_i \vec{a}_i
\]

\[
\sum (r_i \times \vec{F}_i) = \sum (r_i \times m_i \vec{a}_i)
\]

• The system of external forces and the system of effective forces are equipollent by not equivalent.
Vector Mechanics for Engineers: Dynamics

Linear & Angular Momentum

- Linear momentum of the system of particles,
  \[ \mathbf{L} = \sum_{i=1}^{n} m_i \mathbf{v}_i \]
  \[ \dot{\mathbf{L}} = \sum_{i=1}^{n} m_i \dot{\mathbf{v}}_i = \sum_{i=1}^{n} m_i \dot{a}_i \]

- Resultant of the external forces is equal to the rate of change of linear momentum of the system of particles,
  \[ \sum \mathbf{F} = \dot{\mathbf{L}} \]

- Angular momentum about fixed point \( O \) of system of particles,
  \[ \hat{H}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{v}_i) \]
  \[ \dot{\hat{H}}_O = \sum_{i=1}^{n} (\dot{\mathbf{r}}_i \times m_i \mathbf{v}_i) + \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \dot{\mathbf{v}}_i) \]
  \[ = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \dot{a}_i) \]

- Moment resultant about fixed point \( O \) of the external forces is equal to the rate of change of angular momentum of the system of particles,
  \[ \sum \mathbf{M}_O = \dot{\hat{H}}_O \]

Motion of the Mass Center of a System of Particles

- Mass center \( G \) of system of particles is defined by position vector \( \mathbf{r}_G \) which satisfies
  \[ m\mathbf{r}_G = \sum_{i=1}^{n} m_i \mathbf{r}_i \]

- Differentiating twice,
  \[ m\dot{\mathbf{r}}_G = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \]
  \[ m\ddot{r}_G \]
  \[ m\ddot{v}_G = \sum_{i=1}^{n} m_i \ddot{v}_i = \ddot{L} \]
  \[ m\ddot{a}_G = \dot{\mathbf{L}} = \sum \mathbf{F} \]

- The mass center moves as if the entire mass and all of the external forces were concentrated at that point.
• The angular momentum of the system of particles about the mass center,
\[ \mathbf{\hat{H}}_G = \sum_{i=1}^{n} (\mathbf{r}_i' \times m_i \mathbf{v}_i') \]
\[ \mathbf{\hat{H}}_G = \sum_{i=1}^{n} (\mathbf{r}_i' \times m_i \mathbf{a}_i') = \sum_{i=1}^{n} (\mathbf{r}_i' \times m_i (\mathbf{a}_i - \mathbf{a}_G)) \]
\[ = \sum_{i=1}^{n} (\mathbf{r}_i' \times m_i \mathbf{a}_i) - \left( \sum_{i=1}^{n} m_i \mathbf{r}_i' \right) \times \mathbf{a}_G \]
\[ = \sum_{i=1}^{n} (\mathbf{r}_i' \times m_i \mathbf{a}_i) = \sum_{i=1}^{n} (\mathbf{r}_i' \times \mathbf{F}_i) \]
\[ = \sum \mathbf{\bar{M}}_G \]

• Consider the centroidal frame of reference \( Gx' y' z' \), which translates with respect to the Newtonian frame \( Oxyz \).

• The centroidal frame is not, in general, a Newtonian frame.

• The moment resultant about \( G \) of the external forces is equal to the rate of change of angular momentum about \( G \) of the system of particles.

• Angular momentum about \( G \) of particles in their absolute motion relative to the Newtonian \( Oxyz \) frame of reference.
\[ \mathbf{\hat{H}}_G = \sum_{i=1}^{n} (\mathbf{r}_i' \times m_i \mathbf{v}_i) \]
\[ = \sum_{i=1}^{n} (\mathbf{r}_i' \times m_i (\mathbf{v}_G + \mathbf{v}_i')) \]
\[ = \left( \sum_{i=1}^{n} m_i \mathbf{r}_i' \right) \times \mathbf{v}_G + \sum_{i=1}^{n} (\mathbf{r}_i' \times m_i \mathbf{v}_i) \]
\[ \mathbf{\hat{H}}_G = \mathbf{\hat{H}}'_G = \sum \mathbf{\bar{M}}_G \]

• Angular momentum about \( G \) of the particles in their motion relative to the centroidal \( Gx' y' z' \) frame of reference,
\[ \mathbf{\hat{H}}_G = \sum_{i=1}^{n} (\mathbf{r}_i' \times m_i \mathbf{v}_i) \]

• Angular momentum about \( G \) of the particle momenta can be calculated with respect to either the Newtonian or centroidal frames of reference.
Conservation of Momentum

- If no external forces act on the particles of a system, then the linear momentum and angular momentum about the fixed point $O$ are conserved.

\[ \dot{\mathbf{L}} = \sum \mathbf{F} = 0 \quad \dot{\mathbf{H}}_O = \sum \mathbf{M}_O = 0 \]

- Concept of conservation of momentum also applies to the analysis of the mass center motion,

\[ \dot{\mathbf{L}} = \sum \mathbf{F} = 0 \quad \dot{\mathbf{H}}_G = \sum \mathbf{M}_G = 0 \]

\[ \dot{L} = m \dot{v}_G = \text{constant} \quad \dot{H}_G = \text{constant} \]

- In some applications, such as problems involving central forces,

\[ \dot{\mathbf{L}} = \sum \mathbf{F} \neq 0 \quad \dot{\mathbf{H}}_O = \sum \mathbf{M}_O = 0 \]

\[ \dot{L} \neq \text{constant} \quad \dot{H}_O = \text{constant} \]

Sample Problem 14.2

A 20-lb projectile is moving with a velocity of 100 ft/s when it explodes into 5 and 15-lb fragments. Immediately after the explosion, the fragments travel in the directions $\theta_A = 45^\circ$ and $\theta_B = 30^\circ$.

Determine the velocity of each fragment.

SOLUTION:

- Since there are no external forces, the linear momentum of the system is conserved.

- Write separate component equations for the conservation of linear momentum.

- Solve the equations simultaneously for the fragment velocities.
Sample Problem 14.2

Solution:

- Since there are no external forces, the linear momentum of the system is conserved.

\[ \sum (m_i \mathbf{v}_i) = \text{constant} \]

- Write separate component equations for the conservation of linear momentum.

\[
\begin{align*}
    m_A \mathbf{v}_A + m_B \mathbf{v}_B &= m \mathbf{v}_0 \\
    (5/g) \mathbf{v}_A + (15/g) \mathbf{v}_B &= (20/g) \mathbf{v}_0
\end{align*}
\]

- \( x \) components:

\[ 5v_A \cos 45^\circ + 15v_B \cos 30^\circ = 20(100) \]

- \( y \) components:

\[ 5v_A \sin 45^\circ - 15v_B \sin 30^\circ = 0 \]

- Solve the equations simultaneously for the fragment velocities.

\[ v_A = 207 \text{ ft/s} \quad v_B = 97.6 \text{ ft/s} \]

Kinetic Energy

- Kinetic energy of a system of particles,

\[ T = \frac{1}{2} \sum_{i=1}^{n} m_i (\mathbf{v}_i \cdot \mathbf{v}_i) = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 \]

- Expressing the velocity in terms of the centroidal reference frame,

\[ T = \frac{1}{2} \sum_{i=1}^{n} m_i (\mathbf{v}_G + \mathbf{v}_i) \cdot (\mathbf{v}_G + \mathbf{v}_i) \]

\[ = \frac{1}{2} \left( \sum_{i=1}^{n} m_i \mathbf{v}_G^2 + \mathbf{v}_G \cdot \sum_{i=1}^{n} m_i \mathbf{v}_i + \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 \right) \]

\[ = \frac{1}{2} m \mathbf{v}_G^2 + \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 \]

- Kinetic energy is equal to kinetic energy of mass center plus kinetic energy relative to the centroidal frame.
• Principle of work and energy can be applied to each particle $P_i$:
\[ T_1 + U_{1 \rightarrow 2} = T_2 \]
where $U_{1 \rightarrow 2}$ represents the work done by the internal forces $\vec{f}_{ij}$ and the resultant external force $\vec{F}_i$ acting on $P_i$.

• Principle of work and energy can be applied to the entire system by adding the kinetic energies of all particles and considering the work done by all external and internal forces.

• Although $\vec{f}_{ij}$ and $\vec{f}_{ji}$ are equal and opposite, the work of these forces will not, in general, cancel out.

• If the forces acting on the particles are conservative, the work is equal to the change in potential energy and
\[ T_1 + V_1 = T_2 + V_2 \]
which expresses the principle of conservation of energy for the system of particles.

---

Principle of Impulse and Momentum

\[ \sum \vec{F} = \dot{\vec{L}} \]
\[ \sum \int_{t_1}^{t_2} \vec{F} \, dt = \vec{L}_2 - \vec{L}_1 \]
\[ \vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F} \, dt = \vec{L}_2 \]

• The momenta of the particles at time $t_1$ and the impulse of the forces from $t_1$ to $t_2$ form a system of vectors **equipollent** to the system of momenta of the particles at time $t_2$. 

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Sample Problem 14.4

Ball $B$, of mass $m_B$, is suspended from a cord, of length $l$, attached to cart $A$, of mass $m_A$, which can roll freely on a frictionless horizontal tract. While the cart is at rest, the ball is given an initial velocity $v_0 = \sqrt{2gl}$.

Determine (a) the velocity of $B$ as it reaches its maximum elevation, and (b) the maximum vertical distance $h$ through which $B$ will rise.

SOLUTION:

• With no external horizontal forces, it follows from the impulse-momentum principle that the horizontal component of momentum is conserved. This relation can be solved for the velocity of $B$ at its maximum elevation.

• The conservation of energy principle can be applied to relate the initial kinetic energy to the maximum potential energy. The maximum vertical distance is determined from this relation.

V = Vector Mechanics for Engineers: Dynamics

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Sample Problem 14.4

- The conservation of energy principle can be applied to relate the initial kinetic energy to the maximum potential energy.

\[
T_1 + V_1 = T_2 + V_2
\]

Position 1 - Potential Energy: \( V_1 = m_A g l \)

Kinetic Energy: \( T_1 = \frac{1}{2} m_B v_0^2 \)

Position 2 - Potential Energy: \( V_2 = m_A g l + m_B g h \)

Kinetic Energy: \( T_2 = \frac{1}{2} (m_A + m_B) v_{A,2}^2 \)

\[
\frac{1}{2} m_B v_0^2 + m_A g l = \frac{1}{2} (m_A + m_B) v_{A,2}^2 + m_A g l + m_B g h
\]

\[
h = \frac{v_0^2}{2g} - \frac{m_A + m_B}{m_B} \frac{v_{A,2}^2}{2g} = \frac{m_A + m_B}{2g} \frac{v_0^2}{m_B} \left( \frac{m_B}{m_A + m_B} v_0 \right)^2
\]

\[
h = \frac{m_A - v_0^2}{m_A + m_B \frac{v_0}{2g}}
\]

Sample Problem 14.5

Ball A has initial velocity \( v_A = 10 \text{ ft/s} \) parallel to the axis of the table. It hits ball B and then ball C which are both at rest. Balls A and C hit the sides of the table squarely at \( A' \) and \( C' \) and ball B hits obliquely at \( B' \).

Assuming perfectly elastic collisions, determine velocities \( v_A', v_B', v_C' \) with which the balls hit the sides of the table.

SOLUTION:

- There are four unknowns: \( v_A', v_B', v_C' \), and \( v_C \).

- Solution requires four equations: conservation principles for linear momentum (two component equations), angular momentum, and energy.

- Write the conservation equations in terms of the unknown velocities and solve simultaneously.
Sample Problem 14.5

SOLUTION:

- There are four unknowns: \( v_x \), \( v_B, x \), \( v_B, y \), and \( v_C \).

\[
\vec{v}_A = v_A \hat{j} \\
\vec{v}_B = v_{B,x} \hat{i} + v_{B,y} \hat{j} \\
\vec{v}_C = v_C \hat{i}
\]

- The conservation of momentum and energy equations,

\[
\begin{align*}
\dot{L}_1 + \int \vec{F} \, dt &= \dot{L}_2 \\
\dot{m}v_0 &= \dot{m}v_{B,x} + \dot{m}v_{C} = 0 \\
\dot{H}_{O,1} + \int \vec{M} \, dt &= \dot{H}_{O,2} \\
-(2 \text{ ft})mv_0 &= (8 \text{ ft})mv_{A} - (7 \text{ ft})mv_{B,y} - (3 \text{ ft})mv_{C} \\
T_1 + V_1 &= T_2 + V_2 \\
\frac{1}{2}mv_0^2 &= \frac{1}{2}mv_{A}^2 + \frac{1}{2}m(v_{B,x}^2 + v_{B,y}^2) + \frac{1}{2}mv_{C}^2
\end{align*}
\]

Solving the first three equations in terms of \( v_C \),

\[
\begin{align*}
v_A &= v_{B,y} = 3v_C - 20 \\
v_{B,x} &= 10 - v_C
\end{align*}
\]

Substituting into the energy equation,

\[
\begin{align*}
2(3v_C - 20)^2 + (10 - v_C)^2 + v_C^2 &= 100 \\
20v_C^2 - 260v_C + 800 &= 0
\end{align*}
\]

\[
\begin{align*}
v_A &= 4 \text{ ft/s} \\
v_C &= 8 \text{ ft/s} \\
v_B &= 2(4) - 4 = 4 \text{ ft/s}
\end{align*}
\]

\[
\begin{align*}
v_A &= 4 \text{ ft/s} \\
v_B &= 4.47 \text{ ft/s}
\end{align*}
\]