Introduction

- Newton’s first and third laws are sufficient for the study of bodies at rest (statics) or bodies in motion with no acceleration.

- When a body accelerates (changes in velocity magnitude or direction), Newton’s second law is required to relate the motion of the body to the forces acting on it.

- Newton’s second law:
  - A particle will have an acceleration proportional to the magnitude of the resultant force acting on it and in the direction of the resultant force.
  - The resultant of the forces acting on a particle is equal to the rate of change of linear momentum of the particle.
  - The sum of the moments about O of the forces acting on a particle is equal to the rate of change of angular momentum of the particle about O.
Newton’s Second Law of Motion

- Newton’s second law (acceleration):
  - If the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of resultant and in the direction of the resultant.
  - When a particle of mass $m$ is acted upon by a force $\vec{F}$, the acceleration of the particle must satisfy $\vec{F} = m\vec{a}$
  - Acceleration must be evaluated with respect to a Newtonian frame of reference, i.e., one that is not accelerating or rotating.
  - If force acting on particle is zero, particle will not accelerate, i.e., it will remain stationary or continue on a straight line at constant velocity.

Dynamic Equilibrium

- Alternate expression of Newton’s second law,
  \[ \sum \vec{F} - m\vec{a} = 0 \]
  - With the inclusion of the inertial vector, the system of forces acting on the particle is equivalent to zero. The particle is in dynamic equilibrium.
  - Methods developed for particles in static equilibrium may be applied, e.g., coplanar forces may be represented with a closed vector polygon.
  - Inertia vectors are often called inertial forces as they measure the resistance that particles offer to changes in motion, i.e., changes in speed or direction.
  - Inertial forces may be conceptually useful but are not like the contact and gravitational forces found in statics.
Vector Mechanics for Engineers: Dynamics

Linear Momentum of a Particle

- Newton’s second law (linear momentum):
  - The resultant of the forces acting on a particle is equal to the rate of change of linear momentum of the particle.

  \[ \sum \vec{F} = m \frac{d\vec{v}}{dt} \]

  \[ = \frac{d}{dt} (m \vec{v}) = \frac{d\vec{L}}{dt} \]

  \[ \vec{L} = \text{linear momentum of the particle} \]

- Linear Momentum Conservation Principle:
  If the resultant force on a particle is zero, the linear momentum of the particle remains constant in both magnitude and direction.

Equations of Motion

- Newton’s second law provides
  \[ \sum \vec{F} = ma \]

  Solution for particle motion is facilitated by resolving vector equation into scalar component equations, e.g., for rectangular components,

  \[ \sum (F_x \hat{i} + F_y \hat{j} + F_z \hat{k}) = m(a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) \]

  \[ \sum F_x = ma_x \]

  \[ \sum F_y = ma_y \]

  \[ \sum F_z = ma_z \]

  \[ \sum \frac{dv}{dt} = \frac{dy}{dt} \]

  \[ \sum F_n = \frac{v^2}{\rho} \]
For radial and transverse components,

\[
\sum F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)
\]

\[
\sum F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})
\]

Sample Problem 12.1

A 200-lb block rests on a horizontal plane. Find the magnitude of the force \( P \) required to give the block an acceleration or 10 ft/s\(^2\) to the right. The coefficient of kinetic friction between the block and plane is \( \mu_k = 0.25 \).

SOLUTION:

- Resolve the equation of motion for the block into two rectangular component equations.

- Unknowns consist of the applied force \( P \) and the normal reaction \( N \) from the plane. The two equations may be solved for these unknowns.
Sample Problem 12.1

SOLUTION:
• Resolve the equation of motion for the block into two rectangular component equations.

\[ \sum F_x = ma : \]
\[ P \cos 30^\circ - 0.25N = \left(6.21 \text{ lb} \cdot \text{s}^2/\text{ft}\right)10 \text{ ft}/\text{s}^2 \]
\[ = 62.1 \text{ lb} \]
\[ \sum F_y = 0 : \]
\[ N - P \sin 30^\circ - 200 \text{ lb} = 0 \]

• Unknowns consist of the applied force \( P \) and the normal reaction \( N \) from the plane. The two equations may be solved for these unknowns.

\[ N = P \sin 30^\circ + 200 \text{ lb} \]
\[ P \cos 30^\circ - 0.25(P \sin 30^\circ + 200 \text{ lb}) = 62.1 \text{ lb} \]

\[ P = 151 \text{ lb} \]

Sample Problem 12.3

SOLUTION:
• Write the kinematic relationships for the dependent motions and accelerations of the blocks.
• Write the equations of motion for the blocks and pulley.
• Combine the kinematic relationships with the equations of motion to solve for the accelerations and cord tension.

The two blocks shown start from rest. The horizontal plane and the pulley are frictionless, and the pulley is assumed to be of negligible mass. Determine the acceleration of each block and the tension in the cord.
Sample Problem 12.3

**SOLUTION:**

- Write the kinematic relationships for the dependent motions and accelerations of the blocks.
  \[ y_B = \frac{1}{2} x_A, \quad a_B = \frac{1}{2} a_A \]

- Write equations of motion for blocks and pulley.
  \[ \sum F_x = m_A a_A : \]
  \[ T_1 = (100 \text{ kg}) a_A \]
  \[ \sum F_y = m_B a_B : \]
  \[ m_B g - T_2 = m_B a_B \]
  \[ (300 \text{ kg}) (9.81 \text{ m/s}^2) - T_2 = (300 \text{ kg}) a_B \]
  \[ T_2 = 2940 \text{ N} - (300 \text{ kg}) a_B \]
  \[ \sum F_y = m_C a_C = 0 : \]
  \[ T_2 - 2T_1 = 0 \]

**Sample Problem 12.3**

- Combine kinematic relationships with equations of motion to solve for accelerations and cord tension.
  \[ y_B = \frac{1}{2} x_A, \quad a_B = \frac{1}{2} a_A \]
  \[ T_1 = (100 \text{ kg}) a_A \]
  \[ T_2 = 2940 \text{ N} - (300 \text{ kg}) a_B \]
  \[ = 2940 \text{ N} - (300 \text{ kg}) \left( \frac{1}{2} a_A \right) \]
  \[ T_2 - 2T_1 = 0 \]
  \[ 2940 \text{ N} - (150 \text{ kg}) a_A - 2(100 \text{ kg}) a_A = 0 \]

\[ a_A = 8.40 \text{ m/s}^2 \]
\[ a_B = \frac{1}{2} a_A = 4.20 \text{ m/s}^2 \]
\[ T_1 = (100 \text{ kg}) a_A = 840 \text{ N} \]
\[ T_2 = 2T_1 = 1680 \text{ N} \]
Sample Problem 12.4

The 12-lb block \( B \) starts from rest and slides on the 30-lb wedge \( A \), which is supported by a horizontal surface. Neglecting friction, determine (a) the acceleration of the wedge, and (b) the acceleration of the block relative to the wedge.

SOLUTION:

- The block is constrained to slide down the wedge. Therefore, their motions are dependent. Express the acceleration of block as the acceleration of wedge plus the acceleration of the block relative to the wedge.

- Write the equations of motion for the wedge and block.

- Solve for the accelerations.
Sample Problem 12.4

- Solve for the accelerations.
  \[ 0.5N_1 = (W_A/g)\alpha_A \]
  \[ N_1 - W_B \cos 30^\circ = -(W_B/g)\alpha_A \sin 30^\circ \]
  \[ 2(W_A/g)\alpha_A - W_B \cos 30^\circ = -(W_B/g)\alpha_A \sin 30^\circ \]
  \[ a_A = \frac{gW_B \cos 30^\circ}{2W_A + W_B \sin 30^\circ} \]
  \[ a_A = \frac{(32.2 \text{ ft/s}^2)(12 \text{ lb})\cos 30^\circ}{2(30 \text{ lb}) + (12 \text{ lb})\sin 30^\circ} \]
  \[ a_A = 5.07 \text{ ft/s}^2 \]
  \[ a_{B/A} = a_A \cos 30^\circ + g \sin 30^\circ \]
  \[ a_{B/A} = (5.07 \text{ ft/s}^2)\cos 30^\circ + (32.2 \text{ ft/s}^2)\sin 30^\circ \]
  \[ a_{B/A} = 20.5 \text{ ft/s}^2 \]

Sample Problem 12.5

SOLUTION:
- Resolve the equation of motion for the bob into tangential and normal components.
- Solve the component equations for the normal and tangential accelerations.
- Solve for the velocity in terms of the normal acceleration.

The bob of a 2-m pendulum describes an arc of a circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob for the position shown, find the velocity and acceleration of the bob in that position.
Sample Problem 12.5

SOLUTION:
• Resolve the equation of motion for the bob into tangential and normal components.
• Solve the component equations for the normal and tangential accelerations.

\[ \sum F_t = ma_t : \quad mg \sin 30^\circ = ma_t \]
\[ a_t = g \sin 30^\circ \]
\[ a_t = 4.9 \text{ m/s}^2 \]

\[ \sum F_n = ma_n : \quad 2.5mg - mg \cos 30^\circ = ma_n \]
\[ a_n = g(2.5 - \cos 30^\circ) \]
\[ a_n = 16.03 \text{ m/s}^2 \]

• Solve for velocity in terms of normal acceleration.

\[ a_n = \frac{v^2}{\rho} \quad v = \sqrt{\rho a_n} = \sqrt{(2 \text{ m})(16.03 \text{ m/s}^2)} \]
\[ v = \pm 5.66 \text{ m/s} \]

Sample Problem 12.6

Determine the rated speed of a highway curve of radius \( \rho = 400 \text{ ft} \) banked through an angle \( \theta = 18^\circ \). The rated speed of a banked highway curve is the speed at which a car should travel if no lateral friction force is to be exerted at its wheels.

SOLUTION:
• The car travels in a horizontal circular path with a normal component of acceleration directed toward the center of the path. The forces acting on the car are its weight and a normal reaction from the road surface.
  
• Resolve the equation of motion for the car into vertical and normal components.

• Solve for the vehicle speed.
Sample Problem 12.6

SOLUTION:

1. The car travels in a horizontal circular path with a normal component of acceleration directed toward the center of the path. The forces acting on the car are its weight and a normal reaction from the road surface.

2. Resolve the equation of motion for the car into vertical and normal components.

   \[ \sum F_y = 0 : \quad R \cos \theta - W = 0 \]
   
   \[ R = \frac{W}{\cos \theta} \]

   \[ \sum F_n = ma_n : \quad R \sin \theta = \frac{W}{g} a_n \]

   \[ a_n = \frac{W \sin \theta}{g \cos \theta} = \frac{W v^2}{g \rho} \]

3. Solve for the vehicle speed.

   \[ v^2 = g \rho \tan \theta \]
   
   \[ = \left( 32.2 \text{ ft/s}^2 \right) \left( 400 \text{ ft} \right) \tan 18^\circ \]

   \[ v = 64.7 \text{ ft/s} = 44.1 \text{ mi/h} \]

Angular Momentum of a Particle

\[ \vec{H}_O = \vec{r} \times m \vec{\dot{V}} = \text{moment of momentum or the angular momentum of the particle about } O. \]

\[ \vec{H}_O = rm \dot{V} \sin \phi \]

\[ = rm v \dot{\theta} \]

\[ = mr^2 \theta \]

\[ \vec{H}_O = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{pmatrix} \]

\[ \vec{H}_O = \hat{r} \times m \vec{\dot{V}} + \hat{r} \times m \vec{\dot{V}} = \vec{V} \times m \vec{\dot{V}} + \hat{r} \times \vec{m} \vec{\ddot{a}} \]

\[ \vec{H}_O = \vec{r} \times \sum \vec{F} = \sum \vec{M}_O \]
Angular Momentum of a Particle

- Newton’s second law (angular momentum):

\[ \dot{H}_O = \mathbf{r} \times \sum \mathbf{F} = \sum \dot{M}_O \]

- It follows from Newton’s second law that the sum of the moments about \( O \) of the forces acting on the particle is equal to the rate of change of the angular momentum of the particle about \( O \).

Conservation of Angular Momentum

- When only force acting on particle is directed toward or away from a fixed point \( O \), the particle is said to be moving under a central force.

- Since the line of action of the central force passes through \( O \), \( \sum \dot{M}_O = \dot{H}_O = 0 \) and

\[ \mathbf{r} \times m \dot{V} = \dot{H}_O = \text{constant} \]

- Position vector and motion of particle are in a plane perpendicular to \( \dot{H}_O \).

- Magnitude of angular momentum,

\[ H_O = m \dot{V} \sin \phi = \text{constant} \]

\[ = r_0 m V_0 \sin \phi_0 \]

or

\[ H_O = r^2 \dot{\theta} = \text{constant} \]

\[ \frac{H_O}{m} = r^2 \dot{\theta} = h = \frac{\text{angular momentum}}{\text{unit mass}} \]
Conservation of Angular Momentum

- Radius vector $OP$ sweeps infinitesimal area
  
  $dA = \frac{1}{2} r^2 d\theta$

- Define $\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = $ areal velocity

- Recall, for a body moving under a central force, $h = r^2 \dot{\theta} = $ constant

- When a particle moves under a central force, its areal velocity is constant.

Eqs of Motion in Radial & Transverse Components

- Consider particle at $r$ and $\theta$, in polar coordinates,

  $\sum F_r = m a_r = m \left( \ddot{r} - r \dot{\theta}^2 \right)$
  
  $\sum F_\theta = m a_\theta = m \left( r \ddot{\theta} + 2r \dot{\theta} \right)$

- This result may also be derived from conservation of angular momentum,

  $H_O = mr^2 \dot{\theta}$

  $r \sum F_\theta = \frac{d}{dt} (mr^2 \dot{\theta})$

  $= m \left( r^2 \ddot{\theta} + 2r \dot{r} \dot{\theta} \right)$

  $\sum F_\theta = m (r \ddot{\theta} + 2r \dot{\theta})$
A block $B$ of mass $m$ can slide freely on a frictionless arm $OA$ which rotates in a horizontal plane at a constant rate $\dot{\theta}_0$.

Knowing that $B$ is released at a distance $r_0$ from $O$, express as a function of $r$

a) the component $v_r$ of the velocity of $B$ along $OA$, and

b) the magnitude of the horizontal force exerted on $B$ by the arm $OA$.

**SOLUTION:**

- Write the radial and transverse equations of motion for the block.
- Integrate the radial equation to find an expression for the radial velocity.
- Substitute known information into the transverse equation to find an expression for the force on the block.

\[ F = m\dot{\theta}_0^2 \left( r^2 - r_0^2 \right)^{1/2} \]
A satellite is launched in a direction parallel to the surface of the earth with a velocity of 18820 mi/h from an altitude of 240 mi. Determine the velocity of the satellite as it reaches its maximum altitude of 2340 mi. The radius of the earth is 3960 mi.

SOLUTION:
• Since the satellite is moving under a central force, its angular momentum is constant. Equate the angular momentum at A and B and solve for the velocity at B.

\[ rm \sin \phi = H_O = \text{constant} \]

\[ r_A m v_A = r_B m v_B \]

\[ v_B = v_A \frac{r_A}{r_B} \]

\[ = (18820 \text{ mi/h}) \frac{(3960 + 240)\text{ mi}}{(3960 + 2340)\text{ mi}} \]

\[ v_B = 12550 \text{ mi/h} \]
Vector Mechanics for Engineers: Dynamics

% Quiz 1
clear all % clear all variables and breakpoints
t = [0:0.01:1.2];
theta = t.^3+4*t;
r = t.^3+2*t.^2;
x = r.*cos(theta);
y = r.*sin(theta);
v_r = 3*t.^2+4*t;
v_theta = (t.^3+2.*t.^2).*(3.*t.^2+4);
v = sqrt(v_r.^2+v_theta.^2);
a_r = 6*t+4-(t.^3+2.*t.^2).*(3.*t.^2+4).^2;
a_theta = (t.^3+2.*t.^2).*(6*t)+2.*(3*t.^2+4).*(3*t.^2+4*t) ;
a = sqrt(a_r.^2+a_theta.^2);

figure(1)
plot(x,y);
xlabel('x (in.)')
ylabel('y (in.)')
legend('Trajectory',2)
grid on

figure(2)
plot(t,v_r,t,v_theta,t,v)
xlabel('t (sec)')
ylabel('v_r, v_t, v (in./sec)')
legend('v_r','v_t','v',2)
grid on

figure(3)
plot(t,a_r,t,a_theta,t,a)
xlabel('t (sec)')
ylabel('a_r, a_t, a (in./sec^2)')
legend('a_r','a_t','a',2)
grid on
PROBLEM 11.C5

The path of particle P is the ellipse defined by the relations \( r = 1.75(1 - 0.75 \cos \pi t) \) and \( \theta = \pi \), where \( r \) is expressed in inches and \( \theta \) in radians. Derive expressions for the velocity and acceleration of P as a function of \( t \). Consider the time interval \( 0 \leq t \leq 2s \) and plot (a) the components of the velocity and the magnitude of the velocity \( v_r \) and \( v_\theta \) and the magnitude of the velocity \( v \), (b) the components of the acceleration \( a_r \) and \( a_\theta \) and the magnitude of the acceleration \( a \).
% Problem 11.C5
% syms t
theta = pi*t;
r = 1.75/(1-0.75*cos(pi*t));
x = r.*cos(theta);
y = r.*sin(theta);
r_dot = diff(r,t);
r_ddot = diff(r_dot,t);
t_dot = diff(theta,t);
t_ddot = diff(t_dot,t);
v_r = r_dot;
v_theta = r*t_dot;
v = sqrt(v_r.^2+v_theta.^2);
a_r = r_ddot-r*(t_dot)^2;
a_theta = r*t_ddot+2*r_dot*t_dot;
a = sqrt(a_r.^2+a_theta.^2);

t = [0:.01:2];
v_r = subs(v_r,t);
v_theta = subs(v_theta,t);
v = subs(v,t);
a_r = subs(a_r,t);
a_theta = subs(a_theta,t);
a = subs(a,t);
x = subs(x,t);
y = subs(y,t);
PROBLEM 12.C2

A small 0.50-kg block is at rest at the top of a cylindrical surface. The block is given an initial velocity \( \nu_0 \) to the right of magnitude 3 m/s, which causes it to slide on the cylindrical surface. Using computational software calculate and plot the values of \( \theta \) at which the block leaves the surface for values of \( \mu_k \), the coefficient of kinetic friction between the block and the surface, from 0 to 0.4 using 0.05 increments.
function problemc2

fprintf('Mu       Theta\n')
for mk = 0:0.05:0.4
  y0 = [3;0];
  g = 9.81;
  ro = 1.5;
  tspan = [0.275];
  [t,y] = ode45(@f,tspan,y0,[],mk,ro,g);
  eq1 = g*cos(y(:,2)) - y(:,1).^2/ro;
  [x,k] = min(abs(eq1));
  theta = y(k,2)*180/pi;
  fprintf('%-4.2f       %4.2f
',mk,theta);
end

% function dydt = f(t,y,mk,ro,g)
% dydt = [ g*(sin(y(2))-mk*cos(y(2)))+mk*y(1)^2/ro
% y(1)/ro ];

Mu       Theta
0.00     29.59
0.05     30.33
0.10     30.13
0.15     30.84
0.20     31.92
0.25     32.45
0.30     32.56
0.35     32.26
0.40     31.94