Ordinary Differential Equations

Initial Value Problem (IVP)

$$y'' = f(x, y, y') \quad x > 0 \quad \text{with} \quad y(0) = y_0, y'(0) = y'_0$$

Boundary Value Problem (BVP)

$$y'' = f(x, y, y') \quad a < x < b \quad \text{with} \quad y(a) = y_a, y(b) = y_b$$

Initial Value Problem, 1st order

$$y' = f(t, y) \quad t > 0 \quad \text{with} \quad y(0) = y_0$$

forward difference approximation of $y' = \dfrac{y(t+h) - y(t)}{h}$

$$\Rightarrow y(t+h) = y(t) + hf(t, y)$$

Seek for solution in $t_0 \le t \le t_n$ with

$$t_i = t_0 + ih \quad i = 0,1,\ldots,M$$

$$t_{i+1} - t_i = h \quad \text{and} \quad h = \frac{t_n - t_0}{M}$$

$$\Rightarrow y_{i+1} = y_i + hf(t_i, y_i) \quad \text{Euler approximation}$$

Example

$$y' = \frac{t-y}{2} \qquad y(0) = 1$$

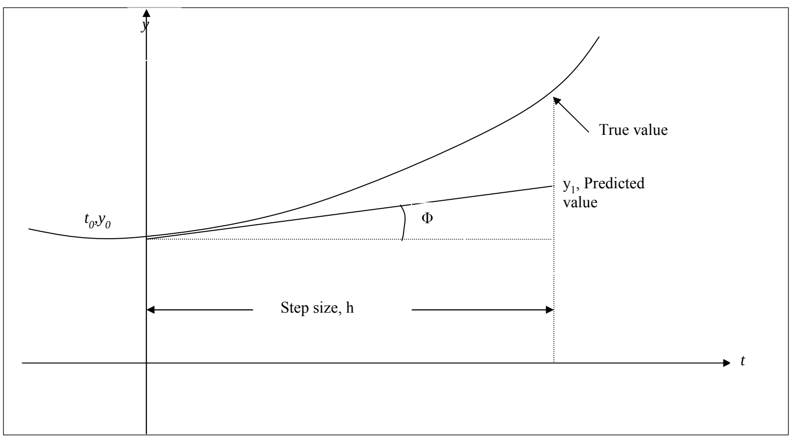
Find $y(t)$ for $0 \le t \le 3$ with $h = 1$

$$y_{i+1} = y_i + hf(t_i, y_i) = y_i + \frac{t_i - y_i}{2} = \frac{t_i + y_i}{2}$$

i	t_i	y_i
0	0	1
1	1	(0+1)/2=0.5
2	2	(1+0.5)/2=0.75
3	3	(0.75+2)/2=1.375

As *h* gets smaller we approach the exact solution.

Geometric interpretation of Euler's Method



$$y' = \frac{t - y}{2} \qquad y(0) = 1$$

Find $y(t)$ **for** $0 \le t \le 3$ **with** $h = 1$

$$y_{i+1} = y_i + hf\,(t_i, y_i) = y_i + \frac{t_i - y_i}{2}$$

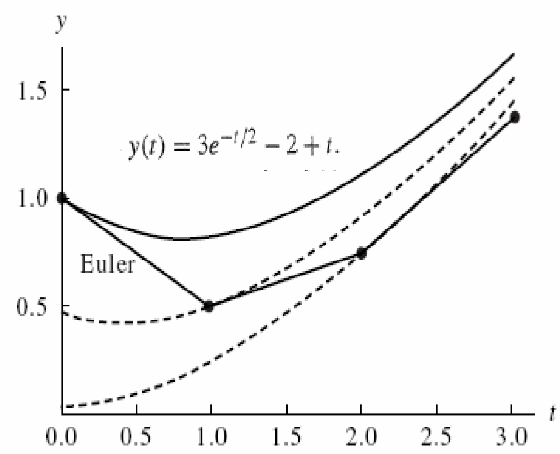


Figure 9.5 Euler's approximations $y_{k-1} = y_k + hf(t_k, y_k)$.

Geometric Description

If you start at the point (t_0, y_0) and compute the value of the slope $m_0 = f(t_0, y_0)$ and move horizontally the amount h and vertically $hf(t_0, y_0)$, then you are moving along the tangent line to $y(t)$ and will end up at the point (t_1, y_1) (see Figure 9.5). Notice that (t_1, y_1) is not on the desired solution curve! But this is the approximation that we are generating. Hence we must use (t_1, y_1) as though it were correct and proceed by computing the slope $m_1 = f(t_1, y_1)$ and using it to obtain the next vertical displacement $hf(t_1, y_1)$ to locate (t_2, y_2), and so on.

Example 9.4. Use Euler's method to solve the I.V.P.

$$y' = \frac{t-y}{2} \quad \text{on } [0, 3] \text{ with } y(0) = 1.$$

Compare solutions for $h = 1, \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$.

Table 9.2 Comparison of Euler Solutions with Different Step Sizes for $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$

t_k	y_k				$y(t_k)$ Exact
	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	
0	1.0	1.0	1.0	1.0	1.0
0.125				0.9375	0.943239
0.25			0.875	0.886719	0.897491
0.375				0.846924	0.862087
0.50		0.75	0.796875	0.817429	0.836402
0.75			0.759766	0.786802	0.811868
1.00	0.5	0.6875	0.758545	0.790158	0.819592
1.50		0.765625	0.846386	0.882855	0.917100
2.00	0.75	0.949219	1.030827	1.068222	1.103638
2.50		1.211914	1.289227	1.325176	1.359514
3.00	1.375	1.533936	1.604252	1.637429	1.669390

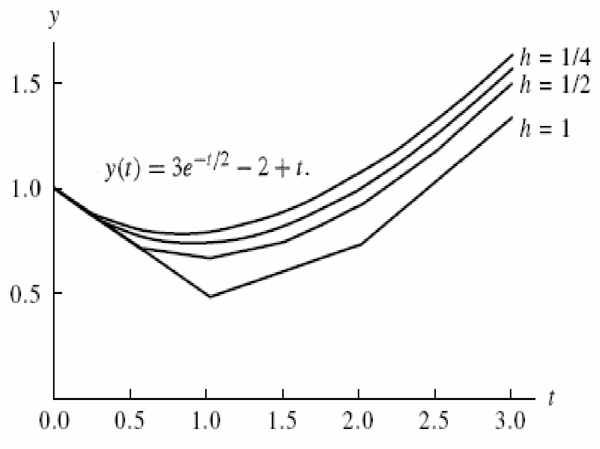


Figure 9.6 Comparison of Euler solutions with different step sizes for $y' = (t - y)/2$ over $[0, 3]$ with the initial condition $y(0) = 1$.

Error in Euler

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(c_1)(t - t_0)^2}{2}.$$

$$y'(t_0) = f(t_0, y(t_0)) \text{ and } h = t_1 - t_0$$

$$y(t_1) = y(t_0) + hf(t_0, y(t_0)) + y''(c_1)\frac{h^2}{2}.$$

$$y_1 = y_0 + hf(t_0, y_0),$$ Euler approximation

$$y''(c_1)\frac{h^2}{2}$$ $\longrightarrow$ Local discretization error

$$\sum_{k=1}^{M} y^{(2)}(c_k)\frac{h^2}{2} \approx My^{(2)}(c)\frac{h^2}{2} = \frac{hM}{2}y^{(2)}(c)h = \frac{(b-a)y^{(2)}(c)}{2}h = O(h^1)$$

$\longrightarrow$ Global discretization error

Final global error: $E(y(b), h) = |y(b) - y_M| = O(h).$

Example 9.5. Compare the F.G.E. when Euler's method is used to solve the I.V.P.

$$y' = \frac{t-y}{2} \quad \text{over } [0, 3] \text{ with } y(0) = 1,$$

using step sizes $1, \frac{1}{2}, \ldots, \frac{1}{64}$.

Table 9.3 Relation between Step Size and F.G.E. for Euler Solutions to $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$

Step size, h	Number of steps, M	Approximation to $y(3)$, y_M	F.G.E. Error at $t = 3$, $y(3) - y_M$	$O(h) \approx Ch$ where $C = 0.256$
1	3	1.375	0.294390	0.256
$\frac{1}{2}$	6	1.533936	0.135454	0.128
$\frac{1}{4}$	12	1.604252	0.065138	0.064
$\frac{1}{8}$	24	1.637429	0.031961	0.032
$\frac{1}{16}$	48	1.653557	0.015833	0.016
$\frac{1}{32}$	96	1.661510	0.007880	0.008
$\frac{1}{64}$	192	1.665459	0.003931	0.004

How to improve accuracy of Euler's Method?

Consider Taylor series

$$y(x+h) = y(x) + hy' + \frac{h^2}{2} y'' + \frac{h^3}{3!} y'''(\eta) + \ldots$$

and compute the derivatives as

$$y'(t) = f$$
$$y''(t) = f_t + f_y y' = f_t + f_y f$$

.....

$$y(x+h) = y(x) + hf + \frac{h^2}{2}\left(f_x + ff_y\right) + \frac{h^3}{3!} y'''(\eta)$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2 f''(x)}{2} + \frac{h^3 f^{(3)}(x)}{6} + \frac{h^4 f^{(4)}(x)}{24} + \cdots$$

For Taylor's formula of order N

Local discretization error = $O(h^{N+1})$

Global discretization error = $O(h^N)$

Example 9.8. Use the Taylor method of order $N = 4$ to solve $y' = (t - y)/2$ on $[0, 3]$ with $y(0) = 1$. Compare solutions for $h = 1, \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$.

$$y(x+h) = y(x) + hy' + \frac{h^2}{2} y^{(2)} + \frac{h^3}{3!} y^{(3)} + \frac{h^4}{4!} y^{(4)}$$

$$y'(t) = \frac{t-y}{2},$$

$$y^{(2)}(t) = \frac{d}{dt}\left(\frac{t-y}{2}\right) = \frac{1-y'}{2} = \frac{1-(t-y)/2}{2} = \frac{2-t+y}{4},$$

$$y^{(3)}(t) = \frac{d}{dt}\left(\frac{2-t+y}{4}\right) = \frac{0-1+y'}{4} = \frac{-1+(t-y)/2}{4} = \frac{-2+t-y}{8},$$

$$y^{(4)}(t) = \frac{d}{dt}\left(\frac{-2+t-y}{8}\right) = \frac{-0+1-y'}{8} = \frac{1-(t-y)/2}{8} = \frac{2-t+y}{16}.$$

Table 9.6 Comparison of the Taylor Solutions of Order $N = 4$ for $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$

	y_k				
t_k	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$y(t_k)$ Exact
0	1.0	1.0	1.0	1.0	1.0
0.125				0.9432392	0.9432392
0.25			0.8974915	0.8974908	0.8974917
0.375				0.8620874	0.8620874
0.50		0.8364258	0.8364037	0.8364024	0.8364023
0.75			0.8118696	0.8118679	0.8118678
1.00	0.8203125	0.8196285	0.8195940	0.8195921	0.8195920
1.50		0.9171423	0.9171021	0.9170998	0.9170997
2.00	1.1045125	1.1036826	1.1036408	1.1036385	1.1036383
2.50		1.3595575	1.3595168	1.3595145	1.3595144
3.00	1.6701860	1.6694308	1.6693928	1.6693906	1.6693905

Final global error: $E(y(3), h) = y(3) - y_M = O(h^4) \approx Ch^4.$

Taylor's method is cumbersome from numerical point of view since higher derivatives need to be calculated.
Alternative way to improve accuracy is to use several function evaluations:

$$y'(t) = f(t, y(t)) \quad \text{over} \quad [a, b] \quad \text{with} \quad y(t_0) = y_0.$$

integrate $y'(t)$ over $[t_0, t_1]$ to get

$$\int_{t_0}^{t_1} f(t, y(t))\, dt = \int_{t_0}^{t_1} y'(t)\, dt = y(t_1) - y(t_0).$$

solved for $y(t_1)$ $\quad y(t_1) = y(t_0) + \int_{t_0}^{t_1} f(t, y(t))\, dt.$

If the trapezoidal rule is used with the step size $h = t_1 - t_0$,

$$y(t_1) \approx y(t_0) + \frac{h}{2}(f(t_0, y(t_0)) + f(t_1, y(t_1))).$$

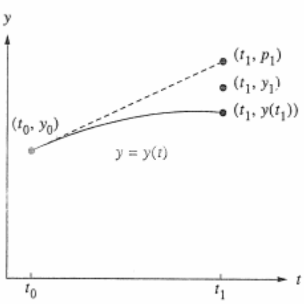
Euler's solution $\quad y(t_1) = y(t_0) + hf(t_0, y(t_0))$

$$y_1 = y(t_0) + \frac{h}{2}(f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0)))$$

$$y_1 = y(t_0) + \frac{h}{2}(f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0))) \quad \textit{Heun's method}$$

↓ slope at the beginning of step

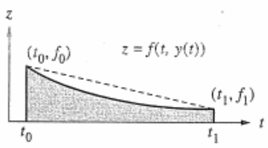
↓ slope at the end of step



(a) Derivative predictor:
$p_1 = y_0 + hf(t_0, y_0)$

$$p_{k+1} = y_k + hf(t_k, y_k), \qquad t_{k+1} = t_k + h,$$

$$y_{k+1} = y_k + \frac{h}{2}(f(t_k, y_k) + f(t_{k+1}, p_{k+1})).$$



(b) Integral corrector:
$y_1 - y_0 = \frac{h}{2}(f_0 + f_1)$

Local DE $\quad -y^{(2)}(c_k)\frac{h^3}{12}.$

Global DE $\quad -\sum_{k=1}^{M} y^{(2)}(c_k)\frac{h^3}{12} \approx \frac{b-a}{12} y^{(2)}(c)h^2 = O(h^2).$

Modified Euler Method
(use two slopes sequentially)

$$y(t+h) = y(t) + hf\left(t + \frac{h}{2}, y + \frac{h}{2} f(t, y)\right)$$

Runge-Kutta Method :
accuracy of Taylor N=4, no high derivatives,
several function evaluations

$$y_{k+1} = y_k + w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4,$$

where k_1, k_2, k_3, and k_4 have the form

$$\begin{aligned} k_1 &= hf(t_k, y_k), \\ k_2 &= hf(t_k + a_1 h, y_k + b_1 k_1), \\ k_3 &= hf(t_k + a_2 h, y_k + b_2 k_1 + b_3 k_2), \\ k_4 &= hf(t_k + a_3 h, y_k + b_4 k_1 + b_5 k_2 + b_6 k_3). \end{aligned}$$

Find a_i, b_i by matching the Runge-Kutta method to N=4 Taylor method.
This results in 11 equations for 13 unknowns.
2 of a_i, b_i are selected and the rest are solved in terms of the selected ones.

the standard Runge-Kutta method of order $N = 4$,

$$y_{k+1} = y_k + \frac{h(f_1 + 2f_2 + 2f_3 + f_4)}{6},$$

where

$$f_1 = f(t_k, y_k),$$
$$f_2 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_1\right),$$
$$f_3 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_2\right),$$
$$f_4 = f(t_k + h, y_k + hf_3).$$

(8) $$y(t_1) - y(t_0) = \int_{t_0}^{t_1} f(t, y(t))\, dt.$$

If Simpson's rule is applied with step size $h/2$, the approximation to the integral in (8) is

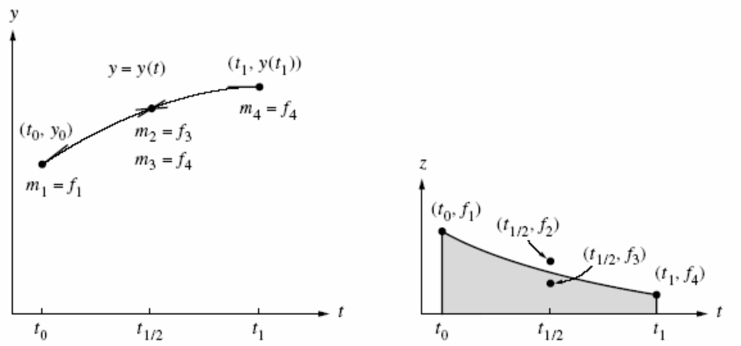
(9) $$\int_{t_0}^{t_1} f(t, y(t))\, dt \approx \frac{h}{6}(f(t_0, y(t_0)) + 4f(t_{1/2}, y(t_{1/2})) + f(t_1, y(t_1))),$$

where $t_{1/2}$ is the midpoint of the interval. Three function values are needed; hence we make the obvious choice $f(t_0, y(t_0)) = f_1$ and $f(t_1, y(t_1)) \approx f_4$. For the value in the middle we chose the average of f_2 and f_3:

$$f(t_{1/2}, y(t_{1/2})) \approx \frac{f_2 + f_3}{2}.$$

These values are substituted into (9), which is used in equation (8) to get y_1:

(10) $$y_1 = y_0 + \frac{h}{6}\left(f_1 + \frac{4(f_2 + f_3)}{2} + f_4\right).$$



(a) Predicted slopes m_j to the solution curve $y = y(t)$

(b) Integral approximation: $y(t_1) - y_0 = \frac{h}{6}(f_1 + 2f_2 + 2f_3 + f_4)$

Figure 9.9 The graphs $y = y(t)$ and $z = f(t, y(t))$ in the discussion of the Runge-Kutta method of order $N = 4$.

error in Simpson ~ $O(h^5)$;
accumulated error in Runge-Kutta after M steps ~ $O(h^4)$

Example 9.11. Compare the F.G.E. when the RK4 method is used to solve $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$ using step sizes $1, \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$.

Table 9.8 Comparison of the RK4 Solutions with Different Step Sizes for $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$

	y_k				
t_k	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$y(t_k)$ Exact
0	1.0	1.0	1.0	1.0	1.0
0.125				0.9432392	0.9432392
0.25			0.8974915	0.8974908	0.8974917
0.375				0.8620874	0.8620874
0.50		0.8364258	0.8364037	0.8364024	0.8364023
0.75			0.8118696	0.8118679	0.8118678
1.00	0.8203125	0.8196285	0.8195940	0.8195921	0.8195920
1.50		0.9171423	0.9171021	0.9170998	0.9170997
2.00	1.1045125	1.1036826	1.1036408	1.1036385	1.1036383
2.50		1.3595575	1.3595168	1.3595145	1.3595144
3.00	1.6701860	1.6694308	1.6693928	1.6693906	1.6693905

Table 9.9 Relation between Step Size and F.G.E. for the RK4 Solutions to $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$

Step size, h	Number of steps, M	Approximation to $y(3)$, y_M	F.G.E. Error at $t = 3$, $y(3) - y_M$	$O(h^4) \approx Ch^4$ where $C = -0.000614$
1	3	1.6701860	−0.0007955	−0.0006140
$\frac{1}{2}$	6	1.6694308	−0.0000403	−0.0000384
$\frac{1}{4}$	12	1.6693928	−0.0000023	−0.0000024
$\frac{1}{8}$	24	1.6693906	−0.0000001	−0.0000001

Remark:

$$y_{k+1} = y_k + w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4,$$

$$\begin{aligned}
k_1 &= hf(t_k, y_k),\\
k_2 &= hf(t_k + a_1 h, y_k + b_1 k_1),\\
k_3 &= hf(t_k + a_2 h, y_k + b_2 k_1 + b_3 k_2),\\
k_4 &= hf(t_k + a_3 h, y_k + b_4 k_1 + b_5 k_2 + b_6 k_3).
\end{aligned}$$

For $k_2, k_3, k_4 = 0$ we recover Euler's method

Runge-Kutta Methods of Order $N = 2$

$$y(t+h) = y(t) + Ahf_0 + Bhf_1,$$

$$\begin{aligned}
f_0 &= f(t, y),\\
f_1 &= f(t + Ph, y + Qhf_0).
\end{aligned}$$

let $f_1 = f(t, y) + Phf_t(t, y) + Qhf_y(t, y)f(t, y) + C_P h^2 + \cdots,$

$$\longrightarrow \quad \begin{aligned} y(t+h) = {} & y(t) + (A+B)hf(t, y) + BPh^2 f_t(t, y)\\ & + BQh^2 f_y(t, y)f(t, y) + BC_P h^3 + \cdots. \end{aligned}$$

Find A, B, P, Q by matching the Runge-Kutta method to N=2 Taylor method:

$$y(t+h) = y(t) + hy'(t) + \frac{1}{2}h^2 y''(t) + C_T h^3 + \cdots,$$

$$\left.\begin{aligned} y'(t) &= f(t, y).\\ y''(t) &= f_t(t, y) + f_y(t, y)y'(t). \end{aligned}\right\} \quad y''(t) = f_t(t, y) + f_y(t, y)f(t, y).$$

$$\longrightarrow \quad \begin{aligned} y(t+h) = {} & y(t) + hf(t, y) + \frac{1}{2}h^2 f_t(t, y)\\ & + \frac{1}{2}h^2 f_y(t, y)f(t, y) + C_T h^3 + \cdots. \end{aligned}$$

$$hf(t,y) = (A+B)hf(t,y) \qquad \text{implies that } 1 = A + B,$$
$$\frac{1}{2}h^2 f_t(t,y) = BPh^2 f_t(t,y) \qquad \text{implies that } \frac{1}{2} = BP,$$
$$\frac{1}{2}h^2 f_y(t,y) f(t,y) = BQh^2 f_y(t,y) f(t,y) \qquad \text{implies that } \frac{1}{2} = BQ.$$

Hence, if we require that A, B, P, and Q satisfy the relations

$$A + B = 1 \qquad BP = \frac{1}{2} \qquad BQ = \frac{1}{2},$$

We need to select one of A,B,P or Q

$$y(t+h) = y(t) + Ahf_0 + Bhf_1,$$

$$f_0 = f(t,y),$$
$$f_1 = f(t + Ph, y + Qhf_0).$$

Case (i): Choose $A = \frac{1}{2}$. This choice leads to $B = \frac{1}{2}$, $P = 1$, and $Q = 1$. If equation (21) is written with these parameters, the formula is

(26) $$y(t+h) = y(t) + \frac{h}{2}(f(t,y) + f(t+h, y + hf(t,y))).$$

When this scheme is used to generate $\{(t_k, y_k)\}$, the result is Heun's method.

Case (ii): Choose $A = 0$. This choice leads to $B = 1$, $P = \frac{1}{2}$, and $Q = \frac{1}{2}$. If equation (21) is written with these parameters, the formula is

(27) $$y(t+h) = y(t) + hf\left(t + \frac{h}{2}, y + \frac{h}{2}f(t,y)\right).$$

When this scheme is used to generate $\{(t_k, y_k)\}$, it is called the ***modified Euler-Cauchy method***.

Sytem of ODEs

$$\frac{dx}{dt} = f(t, x, y) \qquad \text{with} \qquad \begin{cases} x(t_0) = x_0, \\ y(t_0) = y_0. \end{cases}$$
$$\frac{dy}{dt} = g(t, x, y)$$

Seek for solution in $t_0 \le t \le t_n$ with

$$t_i = t_0 + kh \quad k = 0,1,\ldots,M$$

$$t_{k+1} - t_k = h \quad \textbf{and} \quad h = \frac{t_n - t_0}{M}$$

Euler's approximation

$$t_{k+1} = t_k + h,$$
$$x_{k+1} = x_k + hf(t_k, x_k, y_k),$$
$$y_{k+1} = y_k + hg(t_k, x_k, y_k) \qquad \text{for } k = 0, 1, \ldots, M-1.$$

Runge-Kutta method of order=4 (RK4)

$$x_{k+1} = x_k + \frac{h}{6}(f_1 + 2f_2 + 2f_3 + f_4),$$
$$y_{k+1} = y_k + \frac{h}{6}(g_1 + 2g_2 + 2g_3 + g_4),$$

where

$$f_1 = f(t_k, x_k, y_k), \qquad g_1 = g(t_k, x_k, y_k),$$
$$f_2 = f\left(t_k + \frac{h}{2}, x_k + \frac{h}{2}f_1, y_k + \frac{h}{2}g_1\right), \qquad g_2 = g\left(t_k + \frac{h}{2}, x_k + \frac{h}{2}f_1, y_k + \frac{h}{2}g_1\right),$$
$$f_3 = f\left(t_k + \frac{h}{2}, x_k + \frac{h}{2}f_2, y_k + \frac{h}{2}g_2\right), \qquad g_3 = g\left(t_k + \frac{h}{2}, x_k + \frac{h}{2}f_2, y_k + \frac{h}{2}g_2\right),$$
$$f_4 = f(t_k + h, x_k + hf_3, y_k + hg_3), \qquad g_4 = g(t_k + h, x_k + hf_3, y_k + hg_3).$$

Higher order ODEs

$$x''(t) = f(t, x(t), x'(t)) \quad \text{with } x(t_0) = x_0 \text{ and } x'(t_0) = y_0.$$

Reduce the ODE to a system of lower order ODEs

$$x'(t) = y(t). \quad \rightarrow \quad x''(t) = y'(t)$$

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = f(t, x, y) \quad \text{with} \quad \begin{cases} x(t_0) = x_0, \\ y(t_0) = y_0. \end{cases}$$

Example 9.16. Consider the second-order initial value problem

$$x''(t) + 4x'(t) + 5x(t) = 0 \quad \text{with } x(0) = 3 \text{ and } x'(0) = -5.$$

(a) The differential equation has the form

$$x''(t) = f(t, x(t), x'(t)) = -4x'(t) - 5x(t).$$

(b) Using the substitution in (10), we get the reformulated problem:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -5x - 4y \quad \text{with} \quad \begin{cases} x(0) = 3, \\ y(0) = -5. \end{cases}$$

Table 9.14 Runge-Kutta Solution to $x''(t) + 4x'(t) + 5x(t) = 0$ with the Initial Conditions $x(0) = 3$ and $x'(0) = -5$

k	t_k	x_k	$x(t_k)$
0	0.0	3.00000000	3.00000000
1	0.1	2.52564583	2.52565822
2	0.2	2.10402783	2.10404686
3	0.3	1.73506269	1.73508427
4	0.4	1.41653369	1.41655509
5	0.5	1.14488509	1.14490455
10	1.0	0.33324302	0.33324661
20	2.0	−0.00620684	−0.00621162
30	3.0	−0.00701079	−0.00701204
40	4.0	−0.00091163	−0.00091170
48	4.8	−0.00004972	−0.00004969
49	4.9	−0.00002348	−0.00002345
50	5.0	−0.00000493	−0.00000490

Boundary Value Problem (BVP)

$$x'' = f(t, x, x') \quad \text{for } a \le t \le b, \qquad x(a) = \alpha \quad \text{and} \quad x(b) = \beta.$$

Linear BVP

$$x'' = p(t)x'(t) + q(t)\,x(t) + r(t) \quad \text{with } x(a) = \alpha \text{ and } x(b) = \beta$$

Reduction to Two I.V.P.s: Linear Shooting Method

Solution of $x'' = p(t)x'(t) + q(t)x(t) + r(t)$

is given as $x(t) = u(t) + Cv(t)$

where *u* and *v* are the solutions of the following IVPs:

$$u'' = p(t)u'(t) + q(t)u(t) + r(t) \quad \text{with } u(a) = \alpha \text{ and } u'(a) = 0.$$

$$v'' = p(t)v'(t) + q(t)v(t) \quad \text{with } v(a) = 0 \text{ and } v'(a) = 1.$$

proof:

$$\begin{aligned} x'' = u'' + Cv'' &= p(t)u'(t) + q(t)u(t) + r(t) + p(t)Cv'(t) + q(t)Cv(t) \\ &= p(t)(u'(t) + Cv'(t)) + q(t)(u(t) + Cv(t)) + r(t) \\ &= p(t)x'(t) + q(t)x(t) + r(t). \end{aligned}$$

$$x(t) = u(t) + Cv(t)$$

Imposing the boundary condition $x(b) = \beta$

$x(b) = u(b) + Cv(b)$. $\rightarrow$ $C = (\beta - u(b))/v(b)$.

if $v(b) \neq 0$,

$$x(t) = u(t) + \frac{\beta - u(b)}{v(b)} v(t).$$

Example 9.17. Solve the boundary value problem

$$x''(t) = \frac{2t}{1+t^2}x'(t) - \frac{2}{1+t^2}x(t) + 1$$

with $x(0) = 1.25$ and $x(4) = -0.95$ over the interval $[0, 4]$.

Table 9.15 Approximate Solutions $\{x_j\} = \{u_j + w_j\}$ to the Equation $x''(t) = \frac{2t}{1+t^2}x'(t) - \frac{2}{1+t^2} + 1$

t_j	u_j	w_j	$x_j = u_j + w_j$
0.0	1.250000	0.000000	1.250000
0.2	1.220131	0.097177	1.317308
0.4	1.132073	0.194353	1.326426
0.6	0.990122	0.291530	1.281652
0.8	0.800569	0.388707	1.189276
1.0	0.570844	0.485884	1.056728
1.2	0.308850	0.583061	0.891911
1.4	0.022522	0.680237	0.702759
1.6	−0.280424	0.777413	0.496989
1.8	−0.592609	0.874591	0.281982
2.0	−0.907039	0.971767	0.064728
2.2	−1.217121	1.068944	−0.148177
2.4	−1.516639	1.166121	−0.350518
2.6	−1.799740	1.263297	−0.536443
2.8	−2.060904	1.360474	−0.700430
3.0	−2.294916	1.457651	−0.837265
3.2	−2.496842	1.554828	−0.942014
3.4	−2.662004	1.652004	−1.010000
3.6	−2.785960	1.749181	−1.036779
3.8	−2.864481	1.846358	−1.018123
4.0	−2.893535	1.943535	−0.950000

Finite-Difference Method

Consider the linear equation $x'' = p(t)x'(t) + q(t)x(t) + r(t)$

over $[a, b]$ with $x(a) = \alpha$ and $x(b) = \beta$.

The central-difference formulas

$$x'(t_j) = \frac{x(t_{j+1}) - x(t_{j-1})}{2h} + O(h^2)$$

$$x''(t_j) = \frac{x(t_{j+1}) - 2x(t_j) + x(t_{j-1})}{h^2} + O(h^2).$$

$$\frac{x_{j+1} - 2x_j + x_{j-1}}{h^2} = p_j \frac{x_{j+1} - x_{j-1}}{2h} + q_j x_j + r_j,$$

$p_j = p(t_j)$, $q_j = q(t_j)$, and $r_j = r(t_j)$.

$$\left(\frac{-h}{2} p_j - 1\right) x_{j-1} + (2 + h^2 q_j) x_j + \left(\frac{h}{2} p_j - 1\right) x_{j+1} = -h^2 r_j,$$

for $j = 1, 2, \ldots, N-1$, where $x_0 = \alpha$ and $x_N = \beta$.

$$\begin{bmatrix} 2+h^2q_1 & \frac{h}{2}p_1 - 1 & & & \\ \frac{-h}{2}p_2 - 1 & 2+h^2q_2 & \frac{h}{2}p_2 - 1 & & O \\ & \frac{-h}{2}p_j - 1 & 2+h^2q_j & \frac{h}{2}p_j - 1 & \\ O & & \frac{-h}{2}p_{N-2} - 1 & 2+h^2q_{N-2} & \frac{h}{2}p_{N-2} - 1 \\ & & & \frac{-h}{2}p_{N-1} - 1 & 2+h^2q_{N-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_j \\ x_{N-2} \\ x_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} -h^2r_1 + \left(\frac{h}{2}p_1 + 1\right)\alpha \\ -h^2r_2 \\ -h^2r_j \\ -h^2r_{N-2} \\ -h^2r_{N-1} + \left(\frac{-h}{2}p_{N-1} + 1\right)\beta \end{bmatrix}$$

Example 9.18. Solve the boundary value problem

$$x''(t) = \frac{2t}{1+t^2}x'(t) - \frac{2}{1+t^2}x(t) + 1$$

with $x(0) = 1.25$ and $x(4) = -0.95$ over the interval $[0, 4]$.

Table 9.17 Numerical Approximations for $x''(t) = \frac{2t}{1+t^2}x'(t) - \frac{2}{1+t^2}x(t) + 1$

t_j	$x_{j,1}$ $h = 0.2$	$x_{j,2}$ $h = 0.1$	$x_{j,3}$ $h = 0.05$	$x_{j,4}$ $h = 0.025$	$x(t_j)$ exact
0.0	1.250000	1.250000	1.250000	1.250000	1.250000
0.2	1.314503	1.316646	1.317174	1.317306	1.317350
0.4	1.320607	1.325045	1.326141	1.326414	1.326505
0.6	1.272755	1.279533	1.281206	1.281623	1.281762
0.8	1.177399	1.186438	1.188670	1.189227	1.189412
1.0	1.042106	1.053226	1.055973	1.056658	1.056886
1.2	0.874878	0.887823	0.891023	0.891821	0.892086
1.4	0.683712	0.698181	0.701758	0.702650	0.702947
1.6	0.476372	0.492027	0.495900	0.496865	0.497187
1.8	0.260264	0.276749	0.280828	0.281846	0.282184
2.0	0.042399	0.059343	0.063537	0.064583	0.064931
2.2	−0.170616	−0.153592	−0.149378	−0.148327	−0.147977
2.4	−0.372557	−0.355841	−0.351702	−0.350669	−0.350325
2.6	−0.557565	−0.541546	−0.537580	−0.536590	−0.536261
2.8	−0.720114	−0.705188	−0.701492	−0.700570	−0.700262
3.0	−0.854988	−0.841551	−0.838223	−0.837393	−0.837116
3.2	−0.957250	−0.945700	−0.942839	−0.942125	−0.941888
3.4	−1.022221	−1.012958	−1.010662	−1.010090	−1.009899
3.6	−1.045457	−1.038880	−1.037250	−1.036844	−1.036709
3.8	−1.022727	−1.019238	−1.018373	−1.018158	−1.018086
4.0	−0.950000	−0.950000	−0.950000	−0.950000	−0.950000

Table 9.18 Errors in Numerical Approximations Using the Finite-Difference Method

t_j	$x(t_j) - x_{j,1} = e_{j,1}$	$x(t_j) - x_{j,2} = e_{j,2}$	$x(t_j) - x_{j,3} = e_{j,3}$	$x(t_j) - x_{j,4} = e_{j,4}$
	$h_1 = 0.2$	$h_2 = 0.1$	$h_3 = 0.05$	$h_4 = 0.025$
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.002847	0.000704	0.000176	0.000044
0.4	0.005898	0.001460	0.000364	0.000091
0.6	0.009007	0.002229	0.000556	0.000139
0.8	0.012013	0.002974	0.000742	0.000185
1.0	0.014780	0.003660	0.000913	0.000228
1.2	0.017208	0.004263	0.001063	0.000265
1.4	0.019235	0.004766	0.001189	0.000297
1.6	0.020815	0.005160	0.001287	0.000322
1.8	0.021920	0.005435	0.001356	0.000338
2.0	0.022533	0.005588	0.001394	0.000348
2.2	0.022639	0.005615	0.001401	0.000350
2.4	0.022232	0.005516	0.001377	0.000344
2.6	0.021304	0.005285	0.001319	0.000329
2.8	0.019852	0.004926	0.001230	0.000308
3.0	0.017872	0.004435	0.001107	0.000277
3.2	0.015362	0.003812	0.000951	0.000237
3.4	0.012322	0.003059	0.000763	0.000191
3.6	0.008749	0.002171	0.000541	0.000135
3.8	0.004641	0.001152	0.000287	0.000072
4.0	0.000000	0.000000	0.000000	0.000000