# Numerical Integration

## Quadrature

$$\int_a^b f(x)\,dx = \sum_{k=0}^{M} w_k f(x_k) = w_0 f(x_0) + w_1 f(x_1) + \cdots + w_M f(x_M) + E[f]$$

$$a = x_0 < x_1 < \cdots < x_M = b$$

is called a numerical integration or ***quadrature*** formula. The term $E[f]$ is called the ***truncation error*** for integration. The values $\{x_k\}_{k=0}^{M}$ are called the ***quadrature nodes***, and $\{w_k\}_{k=0}^{M}$ are called the ***weights***. ▲

**Definition 7.2.** The ***degree of precision*** of a quadrature formula is the positive integer $n$ such that $E[P_i] = 0$ for all polynomials $P_i(x)$ of degree $i \leq n$, but for which $E[P_{n+1}] \neq 0$ for some polynomial $P_{n+1}(x)$ of degree $n + 1$. ▲

$$E[f] = K f^{(n+1)}(c)$$

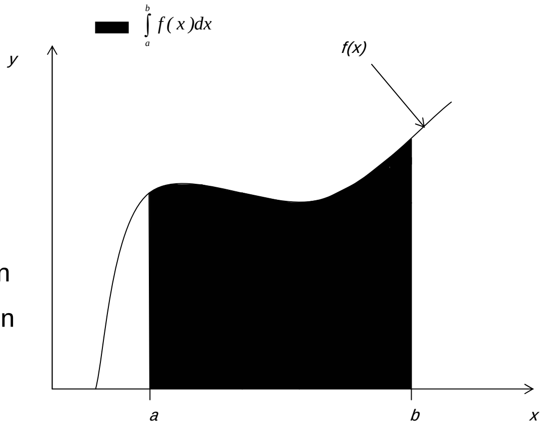
## Newton-Cotes Quadrature

(quadrature based on polynomial interpolation)

**Integration**

The process of measuring the area under a curve.

$$I = \int_a^b f(x)dx$$




Where:

*f(x)* is the integrand

a= lower limit of integration

b= upper limit of integration

## Approximate *f* (*x*) by a straight line

$$\int_{x_0}^{x_1} f(x)\,dx \approx \frac{h}{2}(f_0 + f_1) \qquad \text{(trapezoidal rule),}$$

The trapezoidal rule has degree of precision $n = 1$

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f_0 + f_1) - \frac{h^3}{12} f^{(2)}(c)$$

## Derivation of trapezoidal rule from Lagrange Polynomial

$$f(x) \approx P_M(x) = \sum_{k=0}^{M} f_k L_{M,k}(x),$$

$$\int_{x_0}^{x_M} f(x)\,dx \approx \int_{x_0}^{x_M} P_M(x)\,dx$$

$$P_1(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0} \qquad y_k = f(x_k)$$

$$E_1(x) = \frac{(x - x_0)(x - x_1) f^{(2)}(c)}{2!}$$

Integrate $P_1$ to get trapezoidal rule
Integrate $E_1$ to get the truncation error/degree of precision

## Approximate $f(x)$ by a quadratic curve

$$\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2) \qquad \text{(Simpson's rule)}$$

Simpson's rule has degree of precision $n = 3$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(f_0 + 4f_1 + f_2) - \frac{h^5}{90} f^{(4)}(c)$$

## Derivation of Simpson's rule from Lagrange Polynomial

$$f(x) \approx P_M(x) = \sum_{k=0}^{M} f_k L_{M,k}(x),$$

$$\int_{x_0}^{x_M} f(x)\,dx \approx \int_{x_0}^{x_M} P_M(x)\,dx$$

$$P_2(x) = f_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$E_N(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_N)f^{(N+1)}(c)}{(N+1)!}$$

Integrate $P_2$ to get Simpson's rule
Integrate $E_2$ to get the truncation error/degree of precision

## Approximate $f(x)$ by a cubic curve

$$\int_{x_0}^{x_3} f(x)\,dx \approx \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) \quad \text{(Simpson's } \tfrac{3}{8} \text{ rule)}$$

Simpson's $\frac{3}{8}$ rule has degree of precision $n = 3$

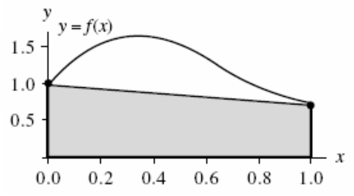
$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80}f^{(4)}(c).$$

**Example 7.2.** Consider the integration of the function $f(x) = 1 + e^{-x}\sin(4x)$ over the fixed interval $[a, b] = [0, 1]$. Apply the various formulas (4) through (7).

For the trapezoidal rule, $h = 1$ and

$$\int_{x_0}^{x_1} f(x)\,dx \approx \frac{h}{2}(f_0 + f_1)$$

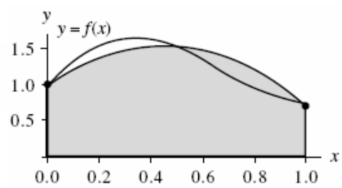
$$\int_0^1 f(x)\,dx \approx \frac{1}{2}(f(0) + f(1)) \quad = 0.86079.$$




For Simpson's rule, $h = 1/2$, and we get

$$\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2)$$

$$\int_0^1 f(x)\,dx \approx \frac{1/2}{3}(f(0) + 4f(\tfrac{1}{2}) + f(1)) = 1.32128$$




For a fair comparison of various methods use the <u>same number of function evaluations</u> in each method.
E.g. Five evaluations in $[x_0, x_4]$

$$\int_{x_0}^{x_4} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \int_{x_3}^{x_4} f(x)\,dx$$

$$(17) \qquad \approx \frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_2) + \frac{h}{2}(f_2 + f_3) + \frac{h}{2}(f_3 + f_4)$$

$$= \frac{h}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + f_4).$$

composite trapezoidal rule

$$\int_{x_0}^{x_4} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx$$

$$(18) \qquad \approx \frac{h}{3}(f_0 + 4f_1 + f_2) + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4).$$

composite Simpson's rule

**Example 7.3.** Consider the integration of the function $f(x) = 1 + e^{-x}\sin(4x)$ over $[a, b] = [0, 1]$. Use exactly five function evaluations and compare the results from the composite trapezoidal rule, composite Simpson rule.

The uniform step size is $h = 1/4$. The composite trapezoidal rule (17) produces

$$\int_0^1 f(x)\,dx \approx \frac{1/4}{2}\left(f(0) + 2f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 2f(\tfrac{3}{4}) + f(1)\right) = 1.28358$$

Using the composite Simpson's rule (18), we get

$$\int_0^1 f(x)\,dx \approx \frac{1/4}{3}\left(f(0) + 4f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 4f(\tfrac{3}{4}) + f(1)\right) = 1.30938.$$

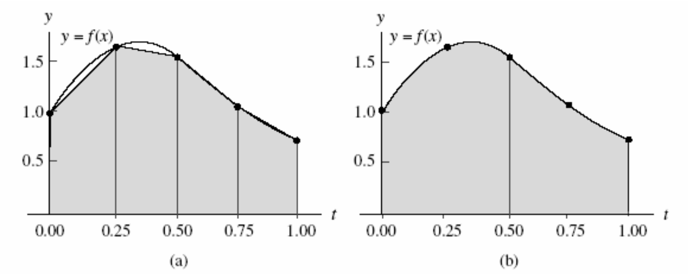


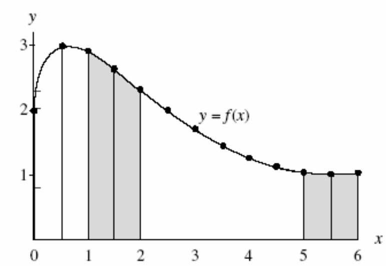

**Figure 7.4** (a) The composite trapezoidal rule yields the approximation 1.28358. (b) The composite Simpson rule yields the approximation 1.30938.

**Theorem 7.2 (Composite Trapezoidal Rule).** Suppose that the interval $[a, b]$ is subdivided into $M$ subintervals $[x_k, x_{k+1}]$ of width $h = (b-a)/M$ by using the equally spaced nodes $x_k = a + kh$, for $k = 0, 1, \ldots, M$. The ***composite trapezoidal rule for M subintervals*** can be expressed

$$\int_a^b f(x)\,dx \approx T(f, h)$$

$$T(f, h) = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + \cdots + 2f_{M-2} + 2f_{M-1} + f_M)$$

$$T(f, h) = \frac{h}{2}\sum_{k=1}^{M}(f(x_{k-1}) + f(x_k))$$



## Trapezoidal Rule: Error Analysis

We first determine the error term when the rule is applied over $[x_0, x_1]$.

$$(10) \quad \int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} P_1(x)\,dx + \int_{x_0}^{x_1} \frac{(x-x_0)(x-x_1)f^{(2)}(c(x))}{2!}\,dx.$$

$$(11) \quad \int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f_0 + f_1) + f^{(2)}(c_1)\int_{x_0}^{x_1} \frac{(x-x_0)(x-x_1)}{2!}\,dx.$$

Use the change of variable $x = x_0 + ht$ in the integral on the right side of (11):

$$(12) \quad \begin{aligned} \int_{x_0}^{x_1} f(x)\,dx &= \frac{h}{2}(f_0 + f_1) + \frac{f^{(2)}(c_1)}{2}\int_0^1 h(t-0)h(t-1)h\,dt \\ &= \frac{h}{2}(f_0 + f_1) + \frac{f^{(2)}(c_1)h^3}{2}\int_0^1 (t^2 - t)\,dt \\ &= \frac{h}{2}(f_0 + f_1) - \frac{f^{(2)}(c_1)h^3}{12}. \end{aligned}$$

Now we are ready to add up the error terms for all of the intervals $[x_k, x_{k+1}]$:

$$(13) \quad \begin{aligned} \int_a^b f(x)\,dx &= \sum_{k=1}^{M} \int_{x_{k-1}}^{x_k} f(x)\,dx \\ &= \sum_{k=1}^{M} \frac{h}{2}(f(x_{k-1}) + f(x_k)) - \frac{h^3}{12}\sum_{k=1}^{M} f^{(2)}(c_k). \end{aligned}$$

The first sum is the composite trapezoidal rule $T(f,h)$. In the second term, one factor of $h$ is replaced with its equivalent $h = (b-a)/M$, and the result is

$$\int_a^b f(x)\,dx = T(f,h) - \frac{(b-a)h^2}{12}\left(\frac{1}{M}\sum_{k=1}^{M} f^{(2)}(c_k)\right).$$

The term in parentheses can be recognized as an average of values for the second derivative and hence is replaced by $f^{(2)}(c)$. Therefore, we have established that

$$\int_a^b f(x)\,dx = T(f,h) - \frac{(b-a)f^{(2)}(c)h^2}{12},$$

**Example 7.7.** Consider $f(x) = 2 + \sin(2\sqrt{x})$. Investigate the error when the composite trapezoidal rule is used over $[1, 6]$ and the number of subintervals is 10, 20, 40, 80, and 160.

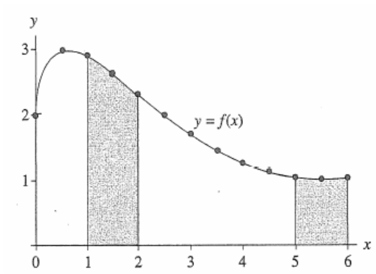
**Table 7.2** Composite Trapezoidal Rule for $f(x) = 2 + \sin(2\sqrt{x})$ over $[1, 6]$

| $M$ | $h$ | $T(f, h)$ | $E_T(f, h) = O(h^2)$ |
|---|---|---|---|
| 10 | 0.5 | 8.19385457 | −0.01037540 |
| 20 | 0.25 | 8.18604926 | −0.00257006 |
| 40 | 0.125 | 8.18412019 | −0.00064098 |
| 80 | 0.0625 | 8.18363936 | −0.00016015 |
| 160 | 0.03125 | 8.18351924 | −0.00004003 |

**Theorem 7.3 (Composite Simpson Rule).** Suppose that $[a, b]$ is subdivided into $2M$ subintervals $[x_k, x_{k+1}]$ of equal width $h = (b-a)/(2M)$ by using $x_k = a + kh$ for $k = 0, 1, \ldots, 2M$. The ***composite Simpson rule for 2M subintervals*** can be expressed

$$\int_a^b f(x)\,dx \approx S(f,h) = \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_{2M})$$

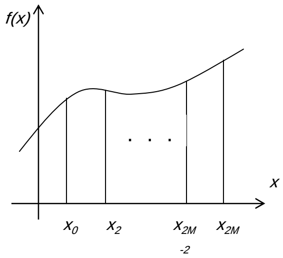
$$= \frac{h}{3}\sum_{k=1}^{M}(f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k}))$$



## Composite Simpson's Rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + .....$$

$$+ \int_{x_{2M-2}}^{x_{2M}} f(x)dx$$




Apply Simpson's Rule over each interval,

$$\int_a^b f(x)dx = (x_2 - x_0)\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + \ldots$$

$$+ (x_{2M} - x_{2M-2})\left[\frac{f(x_{2M-2}) + 4f(x_{2M-1}) + f(x_{2M})}{6}\right]$$

$$x_i - x_{i-2} = 2h$$

## Composite Simpson's Rule

Then

$$\int_a^b f(x)dx = 2h\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + \ldots$$

$$+ 2h\left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}\right] + \ldots$$

$$+ 2h\left[\frac{f(x_{2M-2}) + 4f(x_{2M-1}) + f(x_{2M})}{6}\right]$$

$$\int_a^b f(x)dx = \frac{h}{3}[f(x_0) + 4\{f(x_1) + f(x_3) + \ldots + f(x_{2M-1})\} + \ldots]$$

$$\ldots + 2\{f(x_2) + f(x_4) + \ldots + f(x_{2M-2})\} + f(x_{2M})\}]$$

$$= \frac{h}{3}\left[\sum_{i=1}^{M}\left( f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i}) \right)\right]$$

## Simpson's Rule: Error Analysis

$$\int_a^b f(x)\,dx = S(f,h) + E_S(f,h)$$

$$S(f,h) = \frac{h}{3}(f(a)+f(b)) + \frac{2h}{3}\sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3}\sum_{k=1}^{M} f(x_{2k-1})$$

$$E_S(f,h) = \frac{-(b-a)f^{(4)}(c)h^4}{180}$$

**Table 7.3** Composite Simpson Rule for $f(x) = 2 + \sin(2\sqrt{x})$ over $[1, 6]$

| $M$ | $h$ | $S(f,h)$ | $E_S(f,h) = O(h^4)$ |
|---|---|---|---|
| 5 | 0.5 | 8.18301549 | 0.00046371 |
| 10 | 0.25 | 8.18344750 | 0.00003171 |
| 20 | 0.125 | 8.18347717 | 0.00000204 |
| 40 | 0.0625 | 8.18347908 | 0.00000013 |
| 80 | 0.03125 | 8.18347920 | 0.00000001 |

## Recursive Rules

**Theorem 7.4 (Successive Trapezoidal Rules).** Suppose that $J \geq 1$ and the points $\{x_k = a + kh\}$ subdivide $[a, b]$ into $2^J = 2M$ subintervals of equal width $h = (b-a)/2^J$. The trapezoidal rules $T(f,h)$ and $T(f,2h)$ obey the relationship

(1) $$T(f,h) = \frac{T(f,2h)}{2} + h\sum_{k=1}^{M} f(x_{2k-1}).$$

define $T(J) = T(f,h)$,

$$T(J) = \frac{T(J-1)}{2} + h\sum_{k=1}^{M} f(x_{2k-1})$$

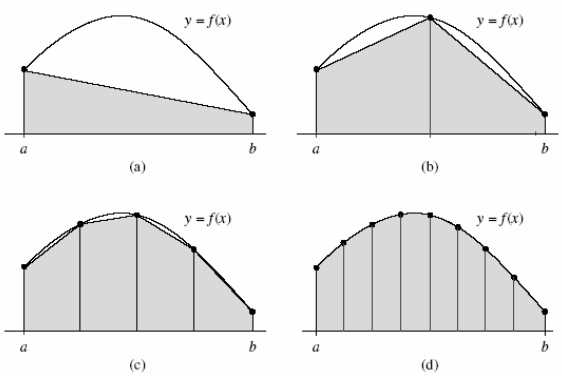



**Figure 7.8** (a) $T(0)$ is the area under $2^0 = 1$ trapezoid. (b) $T(1)$ is the area under $2^1 = 2$ trapezoids. (c) $T(2)$ is the area under $2^2 = 4$ trapezoids. (d) $T(3)$ is the area under $2^3 = 8$ trapezoids.

*Proof.* For the even nodes $x_0 < x_2 < \cdots < x_{2M-2} < x_{2M}$, we use the trapezoidal rule with step size $2h$:

$$(3) \qquad T(J-1) = \frac{2h}{2}(f_0 + 2f_2 + 2f_4 + \cdots + 2f_{2M-4} + 2f_{2M-2} + f_{2M}).$$

For all of the nodes $x_0 < x_1 < x_2 < \cdots < x_{2M-1} < x_{2M}$, we use the trapezoidal rule

with step size $h$:

$$(4) \qquad T(J) = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{2M-2} + 2f_{2M-1} + f_{2M}).$$

Collecting the even and odd subscripts in (4) yields

$$(5) \qquad T(J) = \frac{h}{2}(f_0 + 2f_2 + \cdots + 2f_{2M-2} + f_{2M}) + h\sum_{k=1}^{M} f_{2k-1}.$$

Substituting (3) into (5) results in $T(J) = T(J-1)/2 + h\sum_{k=1}^{M} f_{2k-1}$, and the proof of the theorem is complete. •

**Example 7.11.** Use the sequential trapezoidal rule to compute the approximations $T(0)$, $T(1)$, $T(2)$, and $T(3)$ for the integral $\int_1^5 dx/x = \ln(5) - \ln(1) = 1.609437912$.

Table 7.4 shows the nine values required to compute $T(3)$ and the midpoints required to compute $T(1)$, $T(2)$, and $T(3)$. Details for obtaining the results are as follows:

$$\text{When } h = 4: \quad T(0) = \frac{4}{2}(1.000000 + 0.200000) = 2.400000.$$

$$\text{When } h = 2: \quad T(1) = \frac{T(0)}{2} + 2(0.333333) = 1.200000 + 0.666666 = 1.866666.$$

$$\text{When } h = 1: \quad T(2) = \frac{T(1)}{2} + 1(0.500000 + 0.250000) = 0.933333 + 0.750000 = 1.683333.$$

$$\text{When } h = \frac{1}{2}: \quad T(3) = \frac{T(2)}{2} + \frac{1}{2}(0.666667 + 0.400000 + 0.285714 + 0.222222) = 0.841667 + 0.787302 = 1.628968.$$

■

**Theorem 7.5 (Recursive Simpson Rules).** Suppose that $\{T(J)\}$ is the sequence of trapezoidal rules generated by Corollary 7.4. If $J \geq 1$ and $S(J)$ is Simpson's rule for $2^J$ subintervals of $[a, b]$, then $S(J)$ and the trapezoidal rules $T(J-1)$ and $T(J)$ obey the relationship

$$S(J) = \frac{4T(J) - T(J-1)}{3} \quad \text{for } J = 1, 2, \ldots. \tag{7}$$

*Proof.* The trapezoidal rule $T(J)$ with step size $h$ yields the approximation

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{2M-2} + 2f_{2M-1} + f_{2M}) \\ &= T(J).\end{aligned} \tag{8}$$

The trapezoidal rule $T(J-1)$ with step size $2h$ produces

$$\int_a^b f(x)\,dx \approx h(f_0 + 2f_2 + \cdots + 2f_{2M-2} + f_{2M}) = T(J-1). \tag{9}$$

Multiplying relation (8) by 4 yields

$$\begin{aligned}4\int_a^b f(x)\,dx &\approx h(2f_0 + 4f_1 + 4f_2 + \cdots + 4f_{2M-2} + 4f_{2M-1} + 2f_M) \\ &= 4T(J).\end{aligned} \tag{10}$$

Now subtract (9) from (10) and the result is

$$\begin{aligned}3\int_a^b f(x)\,dx &\approx h(f_0 + 4f_1 + 2f_2 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_{2M}) \\ &= 4T(J) - T(J-1).\end{aligned} \tag{11}$$

This can be rearranged to obtain

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{h}{3}(f_0 + 4f_1 + 2f_2 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_{2M}) \\ &= \frac{4T(J) - T(J-1)}{3}.\end{aligned} \tag{12}$$

The middle term in (12) is Simpson's rule $S(J) = S(f, h)$ and hence the theorem is proved. •

**Example 7.12.** Use the sequential Simpson rule to compute the approximations $S(1)$, $S(2)$, and $S(3)$ for the integral of Example 7.11.

Using the results of Example 7.11 and formula (7) with $J = 1, 2,$ and $3$, we compute

$$S(1) = \frac{4T(1) - T(0)}{3} = \frac{4(1.866666) - 2.400000}{3} = 1.688888,$$
$$S(2) = \frac{4T(2) - T(1)}{3} = \frac{4(1.683333) - 1.866666}{3} = 1.622222,$$
$$S(3) = \frac{4T(3) - T(2)}{3} = \frac{4(1.628968) - 1.683333}{3} = 1.610846. \quad \blacksquare$$

## Romberg Integration

$$S(J) = \frac{4T(J) - T(J-1)}{3} \quad \text{for } J = 1, 2, \ldots.$$

**Definition 7.4.** Define the sequence $\{R(J, K) : J \geq K\}_{J=0}^{\infty}$ of quadrature formulas for $f(x)$ over $[a, b]$ as follows

$$R(J, 0) = T(J) \quad \text{for } J \geq 0, \text{ is the sequential trapezoidal rule.}$$
$$(31) \quad R(J, 1) = S(J) \quad \text{for } J \geq 1, \text{ is the sequential Simpson rule.} \quad \blacktriangle$$

$$R(J, 1) = \frac{4^1 R(J, 0) - R(J-1, 0)}{4^1 - 1} \quad \text{for } J \geq 1$$

recall that

$$\int_a^b f(x)\,dx = T(f, h) + O(h^2),$$
$$\int_a^b f(x)\,dx = S(f, h) + O(h^4),$$

## Romberg Integration

The pattern for integration rules of increasing accuracy is of the form:

$$\int_a^b f(x)\,dx = T(f,h) + O(h^2),$$
$$\int_a^b f(x)\,dx = S(f,h) + O(h^4),$$
$$\int_a^b f(x)\,dx = B(f,h) + O(h^6).$$

$$R(J,0) = T(J)$$
$$R(J,1) = S(J)$$
$$R(J,2) = B(J)$$

$$R(J,1) = \frac{4^1 R(J,0) - R(J-1,0)}{4^1 - 1} \quad \text{for } J \geq 1$$
$$R(J,2) = \frac{4^2 R(J,1) - R(J-1,1)}{4^2 - 1} \quad \text{for } J \geq 2,$$

.....

$$R(J,K) = \frac{4^K R(J,K-1) - R(J-1,K-1)}{4^K - 1} \quad \text{for } J \geq K.$$

## Romberg Integration relies on Richardson's extrapolation

**(Precision of Romberg Integration).** $\int_a^b f(x)\,dx = R(J,K) + O(h^{2K+2}),$

**Lemma 7.1 (Richardson's Improvement for Romberg Integration).** Given two approximations $R(2h, K-1)$ and $R(h, K-1)$ for the quantity $Q$ that satisfy

(28) $$Q = R(h, K-1) + c_1 h^{2K} + c_2 h^{2K+2} + \cdots$$

and

(29) $$Q = R(2h, K-1) + c_1 4^K h^{2K} + c_2 4^{K+1} h^{2K+2} + \cdots,$$

an improved approximation has the form

(30) $$Q = \frac{4^K R(h, K-1) - R(2h, K-1)}{4^K - 1} + O(h^{2K+2}).$$

## Start with $T(f, 2h)$ and $T(f, h)$

(20) $$\int_a^b f(x)\,dx = T(f, 2h) + a_1 4h^2 + a_2 16h^4 + a_3 64h^6 + \cdots$$

(21) $$\int_a^b f(x)\,dx = T(f, h) + a_1 h^2 + a_2 h^4 + a_3 h^6 + \cdots .$$

Multiply equation (21) by 4 and obtain

(22) $$4\int_a^b f(x)\,dx = 4T(f, h) + a_1 4h^2 + a_2 4h^4 + a_3 4h^6 + \cdots .$$

Eliminate $a_1$ by subtracting (20) from (22). The result is

(23) $$3\int_a^b f(x)\,dx = 4T(f, h) - T(f, 2h) - a_2 12h^4 - a_3 60h^6 - \cdots .$$

Now divide equation (23) by 3 and rename the coefficients in the series:

(24) $$\int_a^b f(x)\,dx = \frac{4T(f, h) - T(f, 2h)}{3} + b_1 h^4 + b_2 h^6 + \cdots .$$

$$\Downarrow$$

$$S(f, h)$$

**Table 7.5** Romberg Integration Tableau

| $J$ | $R(J, 0)$ Trapezoidal rule | $R(J, 1)$ Simpson's rule | $R(J, 2)$ Boole's rule | $R(J, 3)$ Third improvement | $R(J, 4)$ Fourth improvement |
|---|---|---|---|---|---|
| 0 | $R(0, 0)$ | | | | |
| 1 | $R(1, 0)$ | $R(1, 1)$ | | | |
| 2 | $R(2, 0)$ | $R(2, 1)$ | $R(2, 2)$ | | |
| 3 | $R(3, 0)$ | $R(3, 1)$ | $R(3, 2)$ | $R(3, 3)$ | |
| 4 | $R(4, 0)$ | $R(4, 1)$ | $R(4, 2)$ | $R(4, 3)$ | $R(4, 4)$ |

**Example 7.14.** Use Romberg integration to find approximations for the definite integral

$$\int_0^{\pi/2} (x^2 + x + 1)\cos(x)\,dx = -2 + \frac{\pi}{2} + \frac{\pi^2}{4} = 2.038197427067\ldots.$$

**Table 7.5** Romberg Integration Tableau

| $J$ | $R(J,0)$ Trapezoidal rule | $R(J,1)$ Simpson's rule | $R(J,2)$ Boole's rule | $R(J,3)$ Third improvement | $R(J,4)$ Fourth improvement |
|---|---|---|---|---|---|
| 0 | $R(0,0)$ | | | | |
| 1 | $R(1,0)$ | $R(1,1)$ | | | |
| 2 | $R(2,0)$ | $R(2,1)$ | $R(2,2)$ | | |
| 3 | $R(3,0)$ | $R(3,1)$ | $R(3,2)$ | $R(3,3)$ | |
| 4 | $R(4,0)$ | $R(4,1)$ | $R(4,2)$ | $R(4,3)$ | $R(4,4)$ |

**Table 7.6** Romberg Integration Tableau for Example 7.14

| $J$ | $R(J,0)$ Trapezoidal rule | $R(J,1)$ Simpson's rule | $R(J,2)$ Boole's rule | $R(J,3)$ Third improvement |
|---|---|---|---|---|
| 0 | 0.785398163397 | | | |
| 1 | 1.726812656758 | 2.040617487878 | | |
| 2 | 1.960534166564 | 2.038441336499 | 2.038296259740 | |
| 3 | 2.018793948078 | 2.038213875249 | 2.038198711166 | 2.038197162776 |
| 4 | 2.033347341805 | 2.038198473047 | 2.038197446234 | 2.038197426156 |
| 5 | 2.036984954990 | 2.038197492719 | 2.038197427363 | 2.038197427064 |

**Example 7.14.** Use Romberg integration to find approximations for the definite integral

$$\int_0^{\pi/2} (x^2 + x + 1)\cos(x)\,dx = -2 + \frac{\pi}{2} + \frac{\pi^2}{4} = 2.038197427067\ldots.$$

**Table 7.6** Romberg Integration Tableau for Example 7.14

| $J$ | $R(J,0)$ Trapezoidal rule | $R(J,1)$ Simpson's rule | $R(J,2)$ Boole's rule | $R(J,3)$ Third improvement |
|---|---|---|---|---|
| 0 | 0.785398163397 | | | |
| 1 | 1.726812656758 | 2.040617487878 | | |
| 2 | 1.960534166564 | 2.038441336499 | 2.038296259740 | |
| 3 | 2.018793948078 | 2.038213875249 | 2.038198711166 | 2.038197162776 |
| 4 | 2.033347341805 | 2.038198473047 | 2.038197446234 | 2.038197426156 |
| 5 | 2.036984954990 | 2.038197492719 | 2.038197427363 | 2.038197427064 |

**Table 7.7** Romberg Error Tableau for Example 7.14

| $J$ | $h$ | $E(J,0) = O(h^2)$ | $E(J,1) = O(h^4)$ | $E(J,2) = O(h^6)$ | $E(J,3) = O(h^8)$ |
|---|---|---|---|---|---|
| 0 | $b-a$ | −1.252799263670 | | | |
| 1 | $\frac{b-a}{2}$ | −0.311384770309 | 0.002420060811 | | |
| 2 | $\frac{b-a}{4}$ | −0.077663260503 | 0.000243909432 | 0.000098832673 | |
| 3 | $\frac{b-a}{8}$ | −0.019403478989 | 0.000016448182 | 0.000001284099 | −0.000000264291 |
| 4 | $\frac{b-a}{16}$ | −0.004850085262 | 0.000001045980 | 0.000000019167 | −0.000000000912 |
| 5 | $\frac{b-a}{32}$ | −0.001212472077 | 0.000000065651 | 0.000000000296 | −0.000000000003 |

# Gauss Quadrature

We wish to find the area under the curve

$$y = f(x), \qquad -1 \leq x \leq 1.$$

What method gives the best answer if only two function evaluations are to be made?

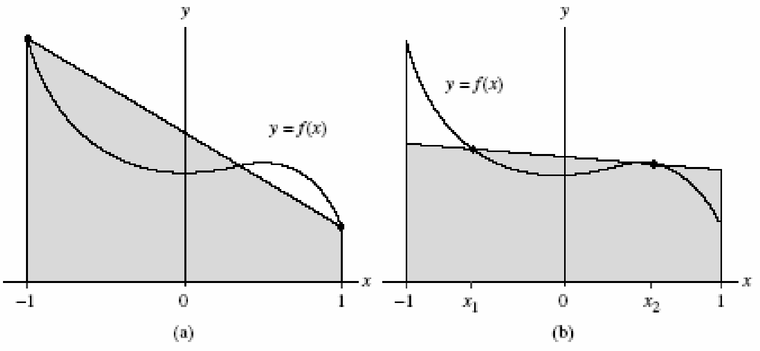



**Figure 7.10** (a) Trapezoidal approximation using the abscissas −1 and 1. (b) Trapezoidal approximation using the abscissas $x_1$ and $x_2$.

# Basis of the Gaussian Quadrature Rule

Previously, the Trapezoidal Rule can be developed by the method of undetermined coefficients as:

$$\int_a^b f(x)dx \cong c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

# Basis of the Gaussian Quadrature Rule

The two-point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the arguments of the function are not predetermined as a and b but as unknowns $x_1$ and $x_2$. In the two-point Gauss Quadrature Rule, the integral is approximated as

$$I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2)$$

To find $w_i$, $x_i$ we need four conditions
= find $w_i$, $x_i$ so that the integration rule has degree of precision 3

$$f(x) = 1: \qquad \int_{-1}^{1} 1\,dx = 2 = w_1 + w_2$$

$$f(x) = x: \qquad \int_{-1}^{1} x\,dx = 0 = w_1 x_1 + w_2 x_2$$

$$f(x) = x^2: \qquad \int_{-1}^{1} x^2\,dx = \frac{2}{3} = w_1 x_1^2 + w_2 x_2^2$$

$$f(x) = x^3: \qquad \int_{-1}^{1} x^3\,dx = 0 = w_1 x_1^3 + w_2 x_2^3.$$

$$w_1 = w_2 = 1.$$

$$-x_1 = x_2 = 1/3^{1/2} \approx 0.5773502692.$$

**(Gauss-Legendre Two-Point Rule).**

$$\int_{-1}^{1} f(x)\,dx \approx G_2(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) + E_2(f),$$

$$E_2(f) = \frac{f^{(4)}(c)}{135}.$$

The Gauss-Legendre rule $G_2(f)$ has degree of precision $n = 3$.

The general $N$-point Gauss-Legendre rule is exact for polynomial functions of degree $\leq 2N-1$, and the numerical integration formula is

(16) $$G_N(f) = w_{N,1}f(x_{N,1}) + w_{N,2}f(x_{N,2}) + \cdots + w_{N,N}f(x_{N,N}).$$

**Table 7.9** Gauss-Legendre Abscissas and Weights

$$\int_{-1}^{1} f(x)\,dx = \sum_{k=1}^{N} w_{N,k}f(x_{N,k}) + E_N(f)$$

| $N$ | Abscissas, $x_{N,k}$ | Weights, $w_{N,k}$ | Truncation error, $E_N(f)$ |
|---|---|---|---|
| 2 | −0.5773502692<br>0.5773502692 | 1.0000000000<br>1.0000000000 | $\frac{f^{(4)}(c)}{135}$ |
| 3 | ±0.7745966692<br>0.0000000000 | 0.5555555556<br>0.8888888888 | $\frac{f^{(6)}(c)}{15{,}750}$ |
| 4 | ±0.8611363116<br>±0.3399810436 | 0.3478548451<br>0.6521451549 | $\frac{f^{(8)}(c)}{3{,}472{,}875}$ |

**Theorem 7.9 (Gauss-Legendre Three-Point Rule).** If $f$ is continuous on $[-1, 1]$, then

(17) $$\int_{-1}^{1} f(x)\,dx \approx G_3(f) = \frac{5f(-\sqrt{3/5}) + 8f(0) + 5f(\sqrt{3/5})}{9}.$$

The Gauss-Legendre rule $G_3(f)$ has degree of precision $n = 5$. If $f \in C^6[-1, 1]$, then

(18) $$\int_{-1}^{1} f(x)\,dx = \frac{5f(-\sqrt{3/5}) + 8f(0) + 5f(\sqrt{3/5})}{9} + E_3(f),$$

where

(19) $$E_3(f) = \frac{f^{(6)}(c)}{15,750}.$$

**Theorem 7.10 (Gauss-Legendre Translation).** Suppose that the abscissas $\{x_{N,k}\}_{k=1}^{N}$ and weights $\{w_{N,k}\}_{k=1}^{N}$ are given for the $N$-point Gauss-Legendre rule over $[-1, 1]$. To apply the rule over the interval $[a, b]$, use the change of variable

$$t = \frac{a+b}{2} + \frac{b-a}{2}x \quad \text{and} \quad dt = \frac{b-a}{2}\,dx. \tag{20}$$

Then the relationship

$$\int_a^b f(t)\,dt = \int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}x\right)\frac{b-a}{2}\,dx \tag{21}$$

is used to obtain the quadrature formula

$$\int_a^b f(t)\,dt = \frac{b-a}{2}\sum_{k=1}^{N} w_{N,k}\, f\left(\frac{a+b}{2} + \frac{b-a}{2}x_{N,k}\right). \tag{22}$$