Numerical Integration

\[ \int_a^b f(x) \, dx = \sum_{k=0}^{M} w_k f(x_k) = w_0 f(x_0) + w_1 f(x_1) + \cdots + w_M f(x_M) + E[f] \]

\( a = x_0 < x_1 < \cdots < x_M = b \)

is called a numerical integration or quadrature formula. The term \( E[f] \) is called the 
truncation error for integration. The values \( \{x_k\}\}_{k=0}^{M} \) are called the quadrature nodes, 
and \( \{w_k\}\}_{k=0}^{M} \) are called the weights.
Definition 7.2. The *degree of precision* of a quadrature formula is the positive integer \( n \) such that \( E[P_i] = 0 \) for all polynomials \( P_i(x) \) of degree \( i \leq n \), but for which \( E[P_{n+1}] \neq 0 \) for some polynomial \( P_{n+1}(x) \) of degree \( n + 1 \).

\[ E[f] = K f^{(n+1)}(c) \]

**Newton-Cotes Quadrature**

(quadature based on polynomial interpolation)
Integration

The process of measuring the area under a curve.

\[ I = \int_{a}^{b} f(x) \, dx \]

Where:
- \( f(x) \) is the integrand
- \( a \) = lower limit of integration
- \( b \) = upper limit of integration

Approximate \( f(x) \) by a straight line

\[ \int_{x_0}^{x_1} f(x) \, dx \approx \frac{h}{2} (f_0 + f_1) \]

(trapezoidal rule)

The trapezoidal rule has degree of precision \( n = 1 \)

\[ \int_{x_0}^{x_1} f(x) \, dx = \frac{h}{2} (f_0 + f_1) - \frac{h^3}{12} f^{(2)}(c) \]
Derivation of trapezoidal rule from Lagrange Polynomial

\[ f(x) \approx P_m(x) = \sum_{k=0}^{M} f_k L_{M,k}(x), \]

\[ \int_{x_0}^{x_1} f(x) \, dx \approx \int_{x_0}^{x_1} P_m(x) \, dx \]

\[ P_1(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0} \quad y_k = f(x_k) \]

\[ E_1(x) = \frac{(x - x_0)(x - x_1)f^{(2)}(c)}{2!} \]

Integrate \( P_1 \) to get trapezoidal rule
Integrate \( E_1 \) to get the truncation error/degree of precision

Approximate \( f(x) \) by a quadratic curve

\[ \int_{x_0}^{x_2} f(x) \, dx \approx \frac{h}{3} \left( f_0 + 4f_1 + f_2 \right) \text{ (Simpson’s rule)} \]

Simpson’s rule has degree of precision \( n = 3 \)

\[ \int_{x_0}^{x_2} f(x) \, dx = \frac{h}{3} \left( f_0 + 4f_1 + f_2 \right) - \frac{h^5}{90} f^{(4)}(c) \]
Derivation of Simpson’s rule from Lagrange Polynomial

\[ f(x) \approx P_M(x) = \sum_{k=0}^{M} f_k L_M(x), \]
\[ \int_{x_0}^{x_M} f(x) \, dx \approx \int_{x_0}^{x_M} P_M(x) \, dx \]

\[ P_2(x) = f_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + f_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \]

\[ E_N(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_N) f^{(N+1)}(c)}{(N+1)!} \]

Integrate \( P_2 \) to get Simpson’s rule.
Integrate \( E_2 \) to get the truncation error/degree of precision.

Approximate \( f(x) \) by a cubic curve

\[ \int_{x_0}^{x_1} f(x) \, dx \approx \frac{3h}{8} (f_0 + 3f_1 + 3f_2 + f_3) \quad \text{(Simpson's \( \frac{3}{8} \) rule)} \]

Simpson’s \( \frac{3}{8} \) rule has degree of precision \( n = 3 \)

\[ \int_{x_0}^{x_1} f(x) \, dx = \frac{3h}{8} (f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80} f^{(4)}(c). \]
For a fair comparison of various methods use the same number of function evaluations in each method. E.g. Five evaluations in \([x_0, x_5]\).

\[
\int_{x_0}^{x_5} f(x) \, dx = \int_{x_0}^{x_1} f(x) \, dx + \int_{x_1}^{x_2} f(x) \, dx + \int_{x_2}^{x_3} f(x) \, dx + \int_{x_3}^{x_4} f(x) \, dx + \int_{x_4}^{x_5} f(x) \, dx
\]

\[
\approx \frac{h}{2} (f_0 + f_1) + \frac{h}{2} (f_1 + f_2) + \frac{h}{2} (f_2 + f_3) + \frac{h}{2} (f_3 + f_4)
\]

\[
= \frac{h}{2} (f_0 + 2f_1 + 2f_2 + 2f_3 + f_4). \quad \text{composite trapezoidal rule}
\]

\[
\int_{x_0}^{x_5} f(x) \, dx = \int_{x_0}^{x_1} f(x) \, dx + \int_{x_1}^{x_2} f(x) \, dx + \int_{x_2}^{x_3} f(x) \, dx + \int_{x_3}^{x_4} f(x) \, dx + \int_{x_4}^{x_5} f(x) \, dx
\]

\[
\approx \frac{h}{2} (f_0 + 4f_1 + f_2) + \frac{h}{3} (f_2 + 4f_3 + f_4)
\]

\[
= \frac{h}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + f_4). \quad \text{composite Simpson’s rule}
\]
Example 7.3. Consider the integration of the function \( f(x) = 1 + e^{-x} \sin(4x) \) over \([a, b] = [0, 1]\). Use exactly five function evaluations and compare the results from the composite trapezoidal rule, composite Simpson rule.

The uniform step size is \( h = \frac{1}{4} \). The composite trapezoidal rule (17) produces

\[
\int_0^1 f(x) \, dx \approx \frac{1}{2} \left( f(0) + 2f(\frac{1}{4}) + 2f(\frac{1}{2}) + 2f(\frac{3}{4}) + f(1) \right) = 1.28358
\]

Using the composite Simpson’s rule (18), we get

\[
\int_0^1 f(x) \, dx \approx \frac{1}{3} \left( f(0) + 4f(\frac{1}{4}) + 2f(\frac{1}{2}) + 4f(\frac{3}{4}) + f(1) \right) = 1.30938.
\]

Figure 7.4 (a) The composite trapezoidal rule yields the approximation 1.28358.
(b) The composite Simpson rule yields the approximation 1.30938.

Theorem 7.2 (Composite Trapezoidal Rule). Suppose that the interval \([a, b]\) is subdivided into \( M \) subintervals \([x_k, x_{k+1}]\) of width \( h = (b-a)/M \) by using the equally spaced nodes \( x_k = a + kh \), for \( k = 0, 1, \ldots, M \). The **composite trapezoidal rule for \( M \) subintervals** can be expressed

\[
\int_a^b f(x) \, dx \approx T(f, h)
\]

\[
T(f, h) = \frac{h}{2} (f_0 + 2f_1 + 2f_2 + 2f_3 + \cdots + 2f_{M-2} + 2f_{M-1} + f_M)
\]

\[
T(f, h) = \frac{h}{2} \sum_{k=1}^{M} (f(x_{k-1}) + f(x_k))
\]

Figure 7.5 (a) The composite trapezoidal rule for \( M = 5 \) subintervals.
Trapezoidal Rule: Error Analysis

We first determine the error term when the rule is applied over \([x_0, x_1]\)

\[
(10) \quad \int_{x_0}^{x_1} f(t) \, dt = \int_{x_0}^{x_1} f(t) \, dt + \int_{x_0}^{x_1} \frac{(x-x_0)(x-x_1)}{2!} f''(c(t)) \, dt.
\]

\[
(11) \quad \int_{x_0}^{x_1} f(x) \, dx = \frac{h}{2} (f_0 + f_1) \int_{x_0}^{x_1} \frac{(x-x_0)(x-x_1)}{2!} \, dx.
\]

Use the change of variable \(x = x_0 + ht\) in the integral on the right side of (11):

\[
(12) \quad \int_{x_0}^{x_1} f(x) \, dx = \frac{h}{2} (f_0 + f_1) \int_{0}^{1} \frac{(x-x_0)(x-x_1)}{2!} \, dx
\]

\[
= \frac{h}{2} (f_0 + f_1) \int_{0}^{1} \frac{x^2 - 1}{12} \, dx
\]

\[
= \frac{h}{2} (f_0 + f_1) \left( \frac{1}{12} \right).
\]

Now we are ready to add up the error terms for all of the intervals \([x_k, x_{k+1}]\):

\[
(13) \quad \int_{a}^{b} f(x) \, dx = \sum_{k=1}^{M} \int_{x_{k-1}}^{x_k} f(x) \, dx
\]

\[
= \sum_{k=1}^{M} \frac{h}{2} (f(x_{k-1}) + f(x_k)) - \frac{h^3}{12} \sum_{k=1}^{M} f''(c_k).
\]

The first sum is the composite trapezoidal rule \(T(f, h)\). In the second term, one factor of \(h\) is replaced with its equivalent \(h = (b - a)/M\), and the result is

\[
\int_{a}^{b} f(x) \, dx = T(f, h) - \frac{(b - a) h^2}{12} \left( \frac{1}{M} \sum_{k=1}^{M} f''(c_k) \right)
\]

The term in parentheses can be recognized as an average of values for the second derivative and hence is replaced by \(f''(c)\). Therefore, we have established that

\[
\int_{a}^{b} f(x) \, dx = T(f, h) - \frac{(b - a) f''(c) h^2}{12}.
\]
Example 7.7. Consider \( f(x) = 2 + \sin(2\sqrt{x}) \). Investigate the error when the composite trapezoidal rule is used over \([1, 6]\) and the number of subintervals is 10, 20, 40, 80, and 160.

<table>
<thead>
<tr>
<th>( M )</th>
<th>( h )</th>
<th>( T(f, h) )</th>
<th>( E_T(f, h) = O(h^2) )</th>
</tr>
</thead>
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<tr>
<td>10</td>
<td>0.5</td>
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<td>-0.01037540</td>
</tr>
<tr>
<td>20</td>
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<td>8.18604926</td>
<td>-0.00257006</td>
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<td>40</td>
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<td>8.18412019</td>
<td>-0.00064098</td>
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<td>8.18363936</td>
<td>-0.00016015</td>
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<td>160</td>
<td>0.03125</td>
<td>8.18351924</td>
<td>-0.00004003</td>
</tr>
</tbody>
</table>

**Theorem 7.3 (Composite Simpson Rule).** Suppose that \([a, b]\) is subdivided into \( 2M \) subintervals \([x_k, x_{k+1}]\) of equal width \( h = (b - a)/(2M) \) by using \( x_k = a + kh \) for \( k = 0, 1, \ldots, 2M \). The composite Simpson rule for \( 2M \) subintervals can be expressed

\[
\int_a^b f(x) \, dx \approx S(f, h) = \frac{h}{3} \left( f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_{2M} \right)
\]

\[
= \frac{h}{3} \sum_{k=1}^{M} \left( f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k}) \right)
\]
Apply Simpson's Rule over each interval,

\[ \int_{x_0}^{x_2} f(x) \, dx + \int_{x_2}^{x_4} f(x) \, dx + \cdots + \int_{x_{2M-2}}^{x_2} f(x) \, dx + \int_{x_0}^{x_2} f(x) \, dx. \]

Apply Simpson's Rule over each interval,

\[ \int_{x_0}^{x_2} f(x) \, dx = (x_2 - x_0) \left[ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + \cdots + (x_{2M} - x_{2M-2}) \left[ \frac{f(x_{2M-2}) + 4f(x_{2M-1}) + f(x_{2M})}{6} \right] \]

\[ x_i - x_{i-2} = 2h \]

Then

\[ \int_{x_0}^{x_2} f(x) \, dx = 2h \left[ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + \cdots + 2h \left[ \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \cdots + 2h \left[ \frac{f(x_{2M-2}) + 4f(x_{2M-1}) + f(x_{2M})}{6} \right] \]

\[ \int_{x_0}^{x_2} f(x) \, dx = \frac{h}{3} \left[ f(x_0) + 4\{f(x_1) + f(x_3) + \cdots + f(x_{2M-1})\} + \cdots \right] \]

\[ \cdots + 2\{f(x_2) + f(x_4) + \cdots + f(x_{2M-2})\} + f(x_{2M}) \]
Simpson’s Rule: Error Analysis

\[
\int_a^b f(x) \, dx = S(f, h) + E_S(f, h)
\]

\[
S(f, h) = \frac{h}{3} (f(a) + f(b)) + \frac{2h}{3} \sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3} \sum_{k=1}^{M} f(x_{2k-1})
\]

\[
E_S(f, h) = \frac{-(b - a) f^{(4)}(c) h^4}{180}
\]

Table 7.3 Composite Simpson Rule for 
\( f(x) = 2 + \sin(2\sqrt{x}) \) over \([1, 6]\)

<table>
<thead>
<tr>
<th>( M )</th>
<th>( h )</th>
<th>( S(f, h) )</th>
<th>( E_S(f, h) = O(h^4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.5</td>
<td>8.18301549</td>
<td>0.00046371</td>
</tr>
<tr>
<td>10</td>
<td>0.25</td>
<td>8.18344750</td>
<td>0.00003171</td>
</tr>
<tr>
<td>20</td>
<td>0.125</td>
<td>8.18347717</td>
<td>0.00000204</td>
</tr>
<tr>
<td>40</td>
<td>0.0625</td>
<td>8.18347908</td>
<td>0.00000013</td>
</tr>
<tr>
<td>80</td>
<td>0.03125</td>
<td>8.18347920</td>
<td>0.00000001</td>
</tr>
</tbody>
</table>

Recursive Rules

**Theorem 7.4 (Successive Trapezoidal Rules).** Suppose that \( J \geq 1 \) and the points \( x_k = a + kh \) subdivide \([a, b]\) into \( 2^J = 2^M \) subintervals of equal width \( h = (b - a)/2^J \). The trapezoidal rules \( T(f, h) \) and \( T(f, 2h) \) obey the relationship

\[
T(f, h) = \frac{T(f, 2h)}{2} + h \sum_{k=1}^{M} f(x_{2k-1}).
\]

![Recursive Rules Diagram](image)

**Figure 7.8** (a) \( T(0) \) is the area under \( 2^0 = 1 \) trapezoid. (b) \( T(1) \) is the area under \( 2^1 = 2 \) trapezoids. (c) \( T(2) \) is the area under \( 2^2 = 4 \) trapezoids. (d) \( T(3) \) is the area under \( 2^3 = 8 \) trapezoids.
Proof. For the even nodes $x_0 < x_2 < \cdots < x_{2M-2} < x_{2M}$, we use the trapezoidal rule with step size $2h$:

\begin{equation}
T(J) = \frac{2h}{2}(f_0 + 2f_2 + 2f_4 + \cdots + 2f_{2M-4} + 2f_{2M-2} + f_{2M}).
\end{equation}

For all of the nodes $x_1 < x_3 < \cdots < x_{2M-1} < x_{2M}$, we use the trapezoidal rule with step size $h$:

\begin{equation}
T(J) = \frac{h}{2}(f_0 + 2f_1 + 2f_3 + \cdots + 2f_{2M-3} + f_{2M-1} + f_{2M}).
\end{equation}

Collecting the even and odd subscripts in (4) yields

\begin{equation}
T(J) = \frac{h}{2}(f_0 + 2f_2 + \cdots + 2f_{2M-2} + f_{2M}) + h \sum_{m=1}^{M} f_{2m-1}.
\end{equation}

Substituting (3) into (5) results in $T(J) = T(J-1)/2 + h \sum_{m=1}^{M} f_{2m-1}$, and the proof of the theorem is complete. 

---

**Example 7.11.** Use the sequential trapezoidal rule to compute the approximations $T(0)$, $T(1)$, $T(2)$, and $T(3)$ for the integral $\int_2^5 dx/x = \ln(5) - \ln(2) = 1.610437912$.

Table 7.4 shows the nine values required to compute $T(1)$ and the midpoints required to compute $T(1)$, $T(2)$, and $T(3)$. Details for obtaining the results are as follows:

When $h = 4$: 
$T(0) = \frac{1}{2} (1.000000 + 0.200000) = 0.600000$.

When $h = 2$: 
$T(1) = \frac{T(0)}{2} + 2(0.333333)$
$= 1.200000 + 0.666666 = 1.866666$.

When $h = 1$: 
$T(2) = \frac{T(1)}{2} + 1(0.500000 + 0.250000)$
$= 0.933333 + 0.750000 = 1.683333$.

When $h = \frac{1}{2}$: 
$T(3) = \frac{T(2)}{2} + \frac{1}{2}(0.666667 + 0.400000 + 0.285714 + 0.222222)$
$= 0.841667 + 0.787302 = 1.628968$. 

---
Theorem 7.5 (Recursive Simpson Rules). Suppose that \( \{T(J)\} \) is the sequence of trapezoidal rules generated by Corollary 7.4. If \( J \geq 1 \) and \( S(J) \) is Simpson’s rule for \( 2^J \) subintervals of \([a, b]\), then \( S(J) \) and the trapezoidal rules \( T(J - 1) \) and \( T(J) \) obey the relationship

\[
S(J) = \frac{4T(J) - T(J - 1)}{3} \quad \text{for } J = 1, 2, \ldots
\]

Proof: The trapezoidal rule \( T(J) \) with step size \( h \) yields the approximation

\[
\int_a^b f(x) \, dx \approx \frac{h}{2} \left( f_0 + 2f_1 + 2f_2 + \cdots + 2f_{2M-2} + 2f_{2M-1} + f_{2M} \right) = T(J).
\]

The trapezoidal rule \( T(J - 1) \) with step size \( 2h \) produces

\[
\int_a^b f(x) \, dx \approx h(f_0 + 2f_2 + \cdots + 2f_{2M-2} + f_{2M}) = T(J - 1).
\]

Multiplying relation (8) by 4 yields

\[
4\int_a^b f(x) \, dx \approx 4f_0 + 4f_2 + \cdots + 4f_{2M-2} + 4f_{2M-1} + 4f_{2M} = 4T(J).
\]

Now subtract (9) from (10) and the result is

\[
3\int_a^b f(x) \, dx \approx h(f_0 + 4f_1 + 2f_2 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_{2M}) = 4T(J) - T(J - 1).
\]

This can be rearranged to obtain

\[
\int_a^b f(x) \, dx \approx \frac{h}{3} \left( f_0 + 4f_1 + 2f_2 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_{2M} \right) = \frac{4T(J) - T(J - 1)}{3}
\]

The middle term in (12) is Simpson’s rule \( S(J) = S(f, h) \) and hence the theorem is proved.
Example 7.12. Use the sequential Simpson rule to compute the approximations $S(1)$, $S(2)$, and $S(3)$ for the integral of Example 7.11.

Using the results of Example 7.11 and formula (7) with $J = 1, 2, 3$, we compute

\[ S(1) = \frac{4T(1) - T(0)}{3} = \frac{4(1.666666) - 2.400000}{3} = 1.688888, \]
\[ S(2) = \frac{4T(2) - T(1)}{3} = \frac{4(1.683333) - 1.866666}{3} = 1.622222, \]
\[ S(3) = \frac{4T(3) - T(2)}{3} = \frac{4(1.628968) - 1.683333}{3} = 1.610846. \]

Romberg Integration

\[ S(J) = \frac{4T(J) - T(J - 1)}{3} \quad \text{for } J = 1, 2, \ldots. \]

Definition 7.4. Define the sequence \( \{R(J, K) : J \geq K \}_{J=0}^{\infty} \) of quadrature formulas for \( f(x) \) over \([a, b]\) as follows

\[ R(J, 0) = T(J) \quad \text{for } J \geq 0, \text{ is the sequential trapezoidal rule}. \]
\[ R(J, 1) = S(J) \quad \text{for } J \geq 1, \text{ is the sequential Simpson rule}. \]

\[ R(J, 1) = \frac{4^J R(J, 0) - R(J - 1, 0)}{4^J - 1} \quad \text{for } J \geq 1 \]

Recall that

\[ \int_a^b f(x) \, dx = T(f, h) + O(h^2), \]
\[ \int_a^b f(x) \, dx = S(f, h) + O(h^4). \]
Romberg Integration

The pattern for integration rules of increasing accuracy is of the form:

\[ \int_a^b f(x) \, dx = T(f, h) + O(h^2), \quad R(J, 0) = T(J) \]
\[ \int_a^b f(x) \, dx = S(f, h) + O(h^4), \quad R(J, 1) = S(J) \]
\[ \int_a^b f(x) \, dx = B(f, h) + O(h^6). \quad R(J, 2) = B(J) \]

\[ R(J, 1) = \frac{4^1 R(J, 0) - R(J - 1, 0)}{4^1 - 1} \quad \text{for } J \geq 1 \]
\[ R(J, 2) = \frac{4^2 R(J, 1) - R(J - 1, 1)}{4^2 - 1} \quad \text{for } J \geq 2, \]

\[ \vdots \]

\[ R(J, K) = \frac{4^K R(J, K - 1) - R(J - 1, K - 1)}{4^K - 1} \quad \text{for } J \geq K. \]

Romberg Integration relies on Richardson’s extrapolation

(Precision of Romberg Integration) \[ \int_a^b f(x) \, dx = R(J, K) + O(h^{2K+2}), \]

Lemma 7.1 (Richardson’s Improvement for Romberg Integration). Given two approximations \( R(2h, K - 1) \) and \( R(h, K - 1) \) for the quantity \( Q \) that satisfy

\[ Q = R(h, K - 1) + c_1 h^{2K} + c_2 h^{2K+2} + \cdots \]

and

\[ Q = R(2h, K - 1) + c_1 4^K h^{2K} + c_2 4^K h^{2K+2} + \cdots , \]

an improved approximation has the form

\[ Q = \frac{4^K R(h, K - 1) - R(2h, K - 1)}{4^K - 1} + O(h^{2K+2}). \]
Start with $T(f, 2h)$ and $T(f, h)$

\[
\int_a^b f(x)\,dx = T(f, 2h) + a_1 h^3 + a_2 4h^4 + a_3 10h^5 + \cdots
\]

\[
\int_a^b f(x)\,dx = T(f, h) + a_1 h + a_2 2h^2 + a_3 3h^3 + \cdots
\]

Multiply equation (21) by 4 and obtain

\[
4\int_a^b f(x)\,dx = 4T(f, h) + a_1 4h^3 + a_2 4h^4 + a_3 4h^5 + \cdots
\]

Eliminate $a_1$ by subtracting (20) from (22). The result is

\[
3\int_a^b f(x)\,dx = 4T(f, h) - T(f, 2h) - a_2 12h^4 - a_3 60h^6 - \cdots
\]

Now divide equation (23) by 3 and rename the coefficients in the series:

\[
\int_a^b f(x)\,dx = \frac{4T(f, h) - T(f, 2h)}{3} + b_1 h^4 + b_2 2h^5 + \cdots
\]

<table>
<thead>
<tr>
<th>$J$</th>
<th>$R(J, 0)$</th>
<th>$R(J, 1)$</th>
<th>$R(J, 2)$</th>
<th>$R(J, 3)$</th>
<th>$R(J, 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trapezoidal rule</td>
<td>Simpson’s rule</td>
<td>Boole’s rule</td>
<td>Third improvement</td>
<td>Fourth improvement</td>
</tr>
<tr>
<td>0</td>
<td>$R(0, 0)$</td>
<td>$R(1, 0)$</td>
<td>$R(2, 0)$</td>
<td>$R(3, 0)$</td>
<td>$R(4, 0)$</td>
</tr>
<tr>
<td>1</td>
<td>$R(1, 0)$</td>
<td>$R(2, 1)$</td>
<td>$R(3, 1)$</td>
<td>$R(4, 1)$</td>
<td>$R(5, 1)$</td>
</tr>
<tr>
<td>2</td>
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<td>$R(3, 1)$</td>
<td>$R(4, 2)$</td>
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<td>$R(6, 2)$</td>
</tr>
<tr>
<td>3</td>
<td>$R(3, 0)$</td>
<td>$R(4, 1)$</td>
<td>$R(5, 2)$</td>
<td>$R(6, 3)$</td>
<td>$R(7, 3)$</td>
</tr>
<tr>
<td>4</td>
<td>$R(4, 0)$</td>
<td>$R(5, 1)$</td>
<td>$R(6, 2)$</td>
<td>$R(7, 3)$</td>
<td>$R(8, 4)$</td>
</tr>
</tbody>
</table>
Example 7.14. Use Romberg integration to find approximations for the definite integral
\[
\int_0^{\pi/2} (x^2 + x + 1) \cos(x) \, dx = -2 + \frac{\pi^2}{2} + \frac{\pi^4}{4} = 2.038197427067 \ldots
\]

Table 7.5 Romberg Integration Tableau

<table>
<thead>
<tr>
<th>J</th>
<th>( R(J, 0) )</th>
<th>( R(J, 1) )</th>
<th>( R(J, 2) )</th>
<th>( R(J, 3) )</th>
<th>( R(J, 4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.785308461397</td>
<td>2.040617458758</td>
<td>2.038296259749</td>
<td>2.0381971462776</td>
<td>2.038197426159</td>
</tr>
<tr>
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<td>2.038441354699</td>
<td>2.03819791166</td>
<td>2.038197426159</td>
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<tr>
<td>2</td>
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<td>2.038197427363</td>
<td>2.038197426159</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.03347341805</td>
<td>2.038197427363</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>4</td>
<td>2.03684954990</td>
<td>2.038197427363</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7.6 Romberg Integration Tableau for Example 7.14

<table>
<thead>
<tr>
<th>J</th>
<th>( R(J, 0) )</th>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 7.14. Use Romberg integration to find approximations for the definite integral
\[
\int_0^{\pi/2} (x^2 + x + 1) \cos(x) \, dx = -2 + \frac{\pi^2}{2} + \frac{\pi^4}{4} = 2.038197427067 \ldots
\]

Table 7.6 Romberg Integration Tableau for Example 7.14

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<tr>
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<th>( R(J, 0) )</th>
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</tr>
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<td>4</td>
<td>2.03684954990</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7.7 Romberg Error Table for Example 7.14

<table>
<thead>
<tr>
<th>J</th>
<th>( h )</th>
<th>( E(J, 0) = O(h^2) )</th>
<th>( E(J, 1) = O(h^3) )</th>
<th>( E(J, 2) = O(h^4) )</th>
<th>( E(J, 3) = O(h^5) )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1/6</td>
<td>-0.252799263670</td>
<td>0.002420098111</td>
<td>0.000000000003273</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1/2</td>
<td>-0.311347702059</td>
<td>0.000000000003273</td>
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<td></td>
</tr>
<tr>
<td>2</td>
<td>1/3</td>
<td>-0.377633265303</td>
<td>0.000000000003273</td>
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<td></td>
</tr>
<tr>
<td>3</td>
<td>1/4</td>
<td>-0.419403479898</td>
<td>0.000000000003273</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1/5</td>
<td>-0.46850885262</td>
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<tr>
<td>5</td>
<td>1/6</td>
<td>-0.51212472077</td>
<td>0.000000000003273</td>
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<td></td>
</tr>
</tbody>
</table>
Gauss Quadrature

We wish to find the area under the curve

\[ y = f(x), \quad -1 \leq x \leq 1. \]

What method gives the best answer if only two function evaluations are to be made?

**Figures 7.10** (a) Trapezoidal approximation using the abscissas $-1$ and $1$. (b) Trapezoidal approximation using the abscissas $x_1$ and $x_2$. 
Basis of the Gaussian Quadrature Rule

Previously, the Trapezoidal Rule can be developed by the method of undetermined coefficients as:

\[ \int_{a}^{b} f(x) \, dx \approx c_1 f(a) + c_2 f(b) \]

where:

\[ c_1 = \frac{b-a}{2} \quad \text{and} \quad c_2 = \frac{b-a}{2} f(b) \]

Basis of the Gaussian Quadrature Rule

The two-point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the arguments of the function are not predetermined as \( a \) and \( b \) but as unknowns \( x_1 \) and \( x_2 \). In the two-point Gauss Quadrature Rule, the integral is approximated as:

\[ I = \int_{a}^{b} f(x) \, dx \approx c_1 f(x_1) + c_2 f(x_2) \]
\[ \int_{-1}^{1} f(x) \, dx \approx w_1 f(x_1) + w_2 f(x_2) \]

To find \( w_i, x_i \) we need four conditions

\[ f(x) = 1: \quad \int_{-1}^{1} 1 \, dx = 2 = w_1 + w_2 \]
\[ f(x) = x: \quad \int_{-1}^{1} x \, dx = 0 = w_1 x_1 + w_2 x_2 \]
\[ f(x) = x^2: \quad \int_{-1}^{1} x^2 \, dx = \frac{2}{3} = w_1 x_1^2 + w_2 x_2^2 \]
\[ f(x) = x^3: \quad \int_{-1}^{1} x^3 \, dx = 0 = w_1 x_1^3 + w_2 x_2^3 . \]

\[ w_1 = w_2 = 1. \]
\[ -x_1 = x_2 = 1/\sqrt{3} \approx 0.5773502692. \]

\( \text{(Gauss-Legendre Two-Point Rule).} \)

\[ \int_{-1}^{1} f(x) \, dx \approx G_2(f) = f \left(\frac{-1}{\sqrt{3}}\right) + f \left(\frac{1}{\sqrt{3}}\right) . \]

\[ \int_{-1}^{1} f(x) \, dx = f \left(\frac{-1}{\sqrt{3}}\right) + f \left(\frac{1}{\sqrt{3}}\right) + E_2(f). \]

\[ E_2(f) = \frac{f^{(4)}(c)}{135} . \]

The Gauss-Legendre rule \( G_2(f) \) has degree of precision \( n = 3 \).
The general $N$-point Gauss-Legendre rule is exact for polynomial functions of degree $\leq 2N - 1$, and the numerical integration formula is

$$G_N(f) = w_{N,1}f(x_{N,1}) + w_{N,2}f(x_{N,2}) + \cdots + w_{N,N}f(x_{N,N}).$$

\begin{table}[h]
\centering
\caption{Gauss-Legendre Abscissas and Weights}
\begin{tabular}{|c|c|c|c|}
\hline
$N$ & Abscissas, $x_{N,i}$ & Weights, $w_{N,i}$ & Truncation error, $E_N(f)$ \\
\hline
2 & -0.5773502692 & 1.0000000000 & $\frac{f^{(2)}(c)}{168}$ \\
 & 0.5773502692 & 1.0000000000 & \\
\hline
3 & $\pm0.774596692$ & 0.5555555556 & $\frac{f^{(4)}(c)}{336}$ \\
 & $\pm0.000000000$ & 0.8888888888 & $18.756$ \\
\hline
4 & $\pm0.841353116$ & 0.3478548451 & $\frac{f^{(6)}(c)}{252}$ \\
 & $\pm0.309016632$ & 0.6521445155 & $3.8752.875$ \\
\hline
\end{tabular}
\end{table}

Theorem 7.9 (Gauss-Legendre Three-Point Rule). If $f$ is continuous on $[-1, 1]$, then

$$\int_{-1}^{1} f(x) \, dx \approx G_3(f) = \frac{5f(-\sqrt{3}/5) + 8f(0) + 5f(\sqrt{3}/5)}{9}.$$ 

The Gauss-Legendre rule $G_3(f)$ has degree of precision $a = 5$. If $f \in C^5[-1, 1]$, then

$$\int_{-1}^{1} f(x) \, dx = \frac{5f(-\sqrt{3}/5) + 8f(0) + 5f(\sqrt{3}/5)}{9} + E_3(f),$$

where

$$E_3(f) = \frac{f^{(6)}(c)}{15}, 750.$$
Theorem 7.10 (Gauss-Legendre Translation). Suppose that the abscissas \( \{x_{N,k}\}_{k=1}^{N} \) and weights \( \{w_{N,k}\}_{k=1}^{N} \) are given for the \( N \)-point Gauss-Legendre rule over \([-1, 1]\). To apply the rule over the interval \([a, b]\), use the change of variable

\[
(20) \quad t = \frac{a+b}{2} + \frac{b-a}{2} x \quad \text{and} \quad dt = \frac{b-a}{2} dx.
\]

Then the relationship

\[
(21) \quad \int_{a}^{b} f(t) \, dt = \int_{-1}^{1} f \left( \frac{a+b}{2} + \frac{b-a}{2} x \right) \frac{b-a}{2} \, dx
\]

is used to obtain the quadrature formula

\[
(22) \quad \int_{a}^{b} f(t) \, dt = \frac{b-a}{2} \sum_{k=1}^{N} w_{N,k} \, f \left( \frac{a+b}{2} + \frac{b-a}{2} x_{N,k} \right).
\]