CHAPTER 11

Introduction

- Rigid Body Dynamics: distance between points is fixed, no size or shape changes, idealization of flexible body
  - particle: mass concentrated at one point, center of mass, rotation of points w/r to c.m. is neglected
  - rigid body: rotation of points w/r to c.m. is taken into account

- Example - Ski jumper:
  - model as a particle: if interested in finding how far she jumps
  - model as a rigid body: if interested in finding the position of the head w/r to torso

- Example - Football player kicking a ball:
  - model as particles: if interested in finding how high/far the ball goes
  - model as rigid bodies: if interested in finding the forces on the knee
Introduction

• Dynamics includes:
  - Kinematics: study of the geometry of motion. Kinematics is used to relate displacement, velocity, acceleration, and time without reference to the cause of motion, i.e. forces are not considered.
  - Kinetics: study of the relations existing between the forces acting on a body, the mass of the body, and the motion of the body. Kinetics is used to predict the motion caused by given forces or to determine the forces required to produce a given motion.

• Newton’s laws:
  1. $F = 0$
  2. $F = ma$
  3. Action-reaction

Derivatives of Functions

• Let $s(u)$ be a scalar function of scalar variable $u$,
  \[ \frac{ds}{du} = \lim_{\Delta u \to 0} \frac{\Delta s}{\Delta u} = \lim_{\Delta u \to 0} \frac{s(u + \Delta u) - s(u)}{\Delta u} \]

• Let $\vec{P}(u)$ be a vector function of scalar variable $u$,
  \[ \frac{d\vec{P}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \vec{P}}{\Delta u} = \lim_{\Delta u \to 0} \frac{\vec{P}(u + \Delta u) - \vec{P}(u)}{\Delta u} \]
Position, Velocity & Acceleration

- Consider a particle moving along a certain path.

- Position vector of a particle at time \( t \) is defined by a vector between origin \( O \) of a fixed reference frame and the position occupied by particle.

- Consider particle which occupies position \( P \) defined by \( \vec{r} \) at time \( t \) and \( P' \) defined by \( \vec{r}' \) at \( t + \Delta t \),

\[
\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \quad \Delta \vec{r} = \vec{r}' - \vec{r}
\]

\[
\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}
\]

- Instantaneous velocity (vector)

- Instantaneous speed (scalar)

- Velocity vector is always tangent to particle path.

- Consider velocity \( \vec{v} \) of particle at time \( t \) and velocity \( \vec{v}' \) at \( t + \Delta t \),

\[
\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}
\]

\[
\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{s}}{\Delta t} = \frac{dv}{dt}
\]

- Instantaneous acceleration (vector)

- In general, acceleration vector is not tangent to particle path.
Motion of a particle along a straight line

- The motion of a particle is known if position is known for all time $t$.
- If path is a straight line we have rectilinear motion, we can describe the motion in terms of $a$, $v$ and $x$.
- Typically, conditions of motion are specified by the type of acceleration experienced by the particle. Determination of velocity and position requires two successive integrations.
- Three classes of motion may be defined for:
  - acceleration given as a function of time, $a = f(t)$
  - acceleration given as a function of position, $a = f(x)$
  - acceleration given as a function of velocity, $a = f(v)$

Acceleration given as a function of time, $a = f(t)$:

$$\frac{dv}{dt} = a = f(t) \quad dv = f(t)dt \quad v(t) - v_0 = \int_0^t f(t)dt$$

$$\frac{dx}{dt} = v(t) \quad dx = v(t)dt \quad x(t) - x_0 = \int_0^t v(t)dt$$

Acceleration given as a function of position, $a = f(x)$:

$$v = \frac{dx}{dt} \quad or \quad dt = \frac{dx}{v} \quad a = \frac{dv}{dt} \quad or \quad a = v \frac{dv}{dx} = f(x)$$

$$vdv = f(x)dx \quad \int_{v_0}^{x} vdv = \int_{x_0}^{x} f(x)dx \quad \frac{1}{2}v(x)^2 - \frac{1}{2}v_0^2 = \int_{x_0}^{x} f(x)dx$$
### Motion of a particle along a straight line

- Acceleration given as a function of velocity, \( a = f(v) \):
  
  \[
  \frac{dv}{dt} = a = f(v) \quad \frac{dv}{f(v)} = dt \quad \int_{v_0}^{v(t)} \frac{dv}{f(v)} = \int_0^t dt
  \]

  \[
  \int_{v_0}^{v(t)} \frac{dv}{f(v)} = t
  \]

  \[
  \frac{v}{dx} = a = f(v) \quad dx = \frac{v}{f(v)} \quad \int_{x_0}^{x(t)} \frac{v}{f(v)} = \int_{v_0}^{v(t)} \frac{v}{f(v)}
  \]

  \[
  x(t) - x_0 = \int_{v_0}^{x(t)} \frac{v}{f(v)}
  \]

### Uniform Rectilinear Motion

For particle in uniform rectilinear motion, the acceleration is zero and the velocity is constant.

\[
\frac{dx}{dt} = v = \text{constant}
\]

\[
\int_{x_0}^{x} dx = \int_0^t dt
\]

\[
x - x_0 = vt
\]

\[
x = x_0 + vt
\]
Uniformly Accelerated Rectilinear Motion

For particle in uniformly accelerated rectilinear motion, the acceleration of the particle is constant.

\[
\frac{dv}{dt} = a = \text{constant} \quad \Rightarrow \quad \int_{v_0}^{v} dv = a \int_{0}^{t} dt \quad \Rightarrow \quad v - v_0 = at
\]

\[
v = v_0 + at
\]

\[
\frac{dx}{dt} = v_0 + at \quad \Rightarrow \quad \int_{x_0}^{x} dx = \int_{0}^{t} (v_0 + at) dt \quad \Rightarrow \quad x - x_0 = v_0 t + \frac{1}{2} at^2
\]

\[
x = x_0 + v_0 t + \frac{1}{2} at^2
\]

\[
\frac{dv}{dx} = a = \text{constant} \quad \Rightarrow \quad \int_{v_0}^{v} dv = a \int_{x_0}^{x} dx \quad \Rightarrow \quad \frac{1}{2} (v^2 - v_0^2) = a (x - x_0)
\]

\[
v^2 = v_0^2 + 2a(x - x_0)
\]

Graphical Solution of Rectilinear-Motion Problems

- Given the \(x-t\) curve, the \(v-t\) curve is equal to the \(x-t\) curve slope.
- Given the \(v-t\) curve, the \(a-t\) curve is equal to the \(v-t\) curve slope.
Graphical Solution of Rectilinear-Motion Problems

- Given the $a-t$ curve, the change in velocity between $t_1$ and $t_2$ is equal to the area under the $a-t$ curve between $t_1$ and $t_2$.

- Given the $v-t$ curve, the change in position between $t_1$ and $t_2$ is equal to the area under the $v-t$ curve between $t_1$ and $t_2$.

Sample Problem 11.2

Determine:

- velocity and elevation above ground at time $t$,
- highest elevation reached by ball and corresponding time, and
- time when ball will hit the ground and corresponding velocity.

Ball tossed with 10 m/s vertical velocity from window 20 m above ground.

SOLUTION:

- Integrate twice to find $v(t)$ and $y(t)$.
- Solve for $t$ at which velocity equals zero (time for maximum elevation) and evaluate corresponding altitude.
- Solve for $t$ at which altitude equals zero (time for ground impact) and evaluate corresponding velocity.
SOLUTION:

• Integrate twice to find \( v(t) \) and \( y(t) \).

\[
\frac{dv}{dt} = a = -9.81 \text{ m/s}^2
\]

\[
v(t) = \int_{t_0}^{t} -9.81 \, dt = v(t) - v_0 = -9.81t
\]

\[
v(t) = 10 \frac{\text{m}}{\text{s}} \left( 9.81 \frac{\text{m}}{\text{s}^2} \right) t
\]

\[
\frac{dy}{dt} = v = 10 - 9.81t
\]

\[
y(t) = \int_{t_0}^{t} (10 - 9.81t) \, dt = y(t) - y_0 = 10t - \frac{1}{2} 9.81 t^2
\]

\[
y(t) = 20 \text{ m} + \left( 10 \frac{\text{m}}{\text{s}} \right) t - \left( 4.905 \frac{\text{m}}{\text{s}^2} \right) t^2
\]

• Solve for \( t \) at which velocity equals zero and evaluate corresponding altitude.

\[
v(t) = 10 \frac{\text{m}}{\text{s}} \left( 9.81 \frac{\text{m}}{\text{s}^2} \right) t = 0
\]

\[
t = 1.019 \text{s}
\]

• Solve for \( t \) at which altitude equals zero and evaluate corresponding velocity.

\[
y(t) = 20 \text{ m} + \left( 10 \frac{\text{m}}{\text{s}} \right) (1.019) - \left( 4.905 \frac{\text{m}}{\text{s}^2} \right) (1.019)^2
\]

\[
y = 25.1 \text{ m}
\]
Sample Problem 11.2

Solve for \( t \) at which altitude equals zero and evaluate corresponding velocity.

\[
y(t) = 20 \text{ m} + \left( 10 \frac{\text{m}}{s} \right) t - \left( 4.905 \frac{\text{m}}{s^2} \right) t^2 = 0
\]

\[
t = 3.28 \text{ s}
\]

\[
\nu(t) = 10 \frac{\text{m}}{s} - \left( 9.81 \frac{\text{m}}{s^2} \right) t
\]

\[
\nu(3.28) = 10 \frac{\text{m}}{s} - \left( 9.81 \frac{\text{m}}{s^2} \right) (3.28) \text{ s} = -22.2 \frac{\text{m}}{s}
\]

Sample Problem 11.3

Brake mechanism used to reduce gun recoil consists of piston attached to barrel moving in fixed cylinder filled with oil. As barrel recoils with initial velocity \( v_0 \), piston moves and oil is forced through orifices in piston, causing piston and cylinder to decelerate at rate proportional to their velocity.

Determine \( v(t) \), \( x(t) \), and \( v(x) \).

SOLUTION:

• Integrate \( a = \frac{dv}{dt} = -kv \) to find \( v(t) \).

• Integrate \( v(t) = \frac{dx}{dt} \) to find \( x(t) \).

• Integrate \( a = v \frac{dv}{dx} = -kv \) to find \( v(x) \).
Sample Problem 11.3

SOLUTION:

- Integrate $a = \frac{dv}{dt} = -kv$ to find $v(t)$.

$$a = \frac{dv}{dt} = -kv \quad v(t) = \frac{dv}{v} = -k \int_0^t dt \quad \ln \frac{v(t)}{v_0} = -kt$$

$$v(t) = v_0 e^{-kt}$$

- Integrate $v(t) = \frac{dx}{dt}$ to find $x(t)$.

$$v(t) = \frac{dx}{dt} = v_0 e^{-kt}$$

$$x(t) = \int_0^t v_0 e^{-kt} dt = v_0 \left[ -\frac{1}{k} e^{-kt} \right]_0^t$$

$$x(t) = \frac{v_0}{k} \left( 1 - e^{-kt} \right)$$

Alternatively,

$$v(t) = v_0 e^{-kt}$$

with $$x(t) = \frac{v_0}{k} \left[ 1 - e^{-kt} \right]$$

and $$v(t) = v_0 e^{-kt} \quad \text{or} \quad e^{-kt} = \frac{v(t)}{v_0}$$

then $$x(t) = \frac{v_0}{k} \left( 1 - \frac{v(t)}{v_0} \right)$$

$$v = v_0 - kx$$
Motion of Several Particles: Dependent Motion

- Position of a particle may depend on position of one or more other particles.
- Position of block B depends on position of block A. Since rope is of constant length, it follows that sum of lengths of segments must be constant.
  \[ x_A + 2x_B = \text{constant} \] (one degree of freedom)
- Positions of three blocks are dependent.
  \[ 2x_A + x_B + x_C = \text{constant} \] (two degrees of freedom)
- For linearly related positions, similar relations hold between velocities and accelerations.
  \[
  2 \frac{dx_A}{dt} + 2 \frac{dx_B}{dt} + \frac{dx_C}{dt} = 0 \quad \text{or} \quad 2v_A + 2v_B + v_C = 0
  \]
  \[
  2 \frac{dv_A}{dt} + 2 \frac{dv_B}{dt} + \frac{dv_C}{dt} = 0 \quad \text{or} \quad 2a_A + 2a_B + a_C = 0
  \]

Sample Problem 11.5

Pulley D is attached to a collar which is pulled down at 3 in./s. At \( t = 0 \), collar A starts moving down from K with constant acceleration and zero initial velocity. Knowing that velocity of collar A is 12 in./s as it passes L, determine the change in elevation, velocity, and acceleration of block B when block A is at L.

SOLUTION:
- Define origin at upper horizontal surface with positive displacement downward.
- Collar A has uniformly accelerated rectilinear motion. Solve for acceleration and time \( t \) to reach L.
- Pulley D has uniform rectilinear motion. Calculate change of position at time \( t \).
- Block B motion is dependent on motions of collar A and pulley D. Write motion relationship and solve for change of block B position at time \( t \).
- Differentiate motion relation twice to develop equations for velocity and acceleration of block B.
Sample Problem 11.5

SOLUTION:

- Define origin at upper horizontal surface with positive displacement downward.

- Collar A has uniformly accelerated rectilinear motion. Solve for acceleration and time $t$ to reach $L$.

\[
\begin{align*}
v_A^2 &= (v_A)_0^2 + 2a_A(x_A - (x_A)_0) \\
\left(\frac{12 \text{ in.}}{s}\right)^2 &= 2a_A(8 \text{ in.}) \\
a_A &= \frac{9 \text{ in.}}{s^2}
\end{align*}
\]

\[
\begin{align*}
v_A &= (v_A)_0 + a_A t \\
12 \text{ in.} &= 9 \text{ in./s}^2 t \\
t &= 1.333 \text{ s}
\end{align*}
\]

- Pulley D has uniform rectilinear motion. Calculate change of position at time $t$.

\[
x_D = (x_D)_0 + v_D t \\
x_D - (x_D)_0 = \left(3 \frac{\text{in.}}{s}\right)(1.333 \text{ s}) = 4 \text{ in.}
\]

- Block B motion is dependent on motions of collar A and pulley D. Write motion relationship and solve for change of block B position at time $t$.

Total length of cable remains constant,

\[
\begin{align*}
x_A + 2x_D + x_B &= (x_A)_0 + 2(x_D)_0 + (x_B)_0 \\
\left[x_A - (x_A)_0\right] + 2\left[x_D - (x_D)_0\right] + [x_B - (x_B)_0] &= 0 \\
(8 \text{ in.}) + 2(4 \text{ in.}) + [x_B - (x_B)_0] &= 0 \\
x_B - (x_B)_0 &= -16 \text{ in.}
\end{align*}
\]
Sample Problem 11.5

- Differentiate motion relation twice to develop equations for velocity and acceleration of block B.

\[ x_A + 2x_D + x_B = \text{constant} \]

\[ v_A + 2v_D + v_B = 0 \]

\[ \begin{align*}
12 \text{ in.}/\text{s} + 2 \left( \frac{3 \text{ in.}}{\text{s}} \right) + v_B &= 0 \\
18 \text{ in.}/\text{s}^2 + v_B &= 0
\end{align*} \]

\[ v_B = 18 \text{ in.}/\text{s} \]

\[ a_A + 2a_D + a_B = 0 \]

\[ \begin{align*}
9 \text{ in.}^2/\text{s}^2 + v_B &= 0 \\
a_B &= -9 \text{ in.}^2/\text{s}^2
\end{align*} \]

Rectangular Components of Velocity & Acceleration

- When position vector of particle \( P \) is given by its rectangular components,

\[ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \]

- Velocity vector,

\[ \vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} \]

\[ \ddot{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \]

- Acceleration vector,

\[ \ddot{a} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k} = \dddot{x}\hat{i} + \dddot{y}\hat{j} + \dddot{z}\hat{k} \]

\[ \dddot{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \]
Rectangular Components of Velocity & Acceleration

- Rectangular components particularly effective when component accelerations can be integrated independently, e.g., motion of a projectile,
  \[ a_x = \ddot{x} = 0 \quad a_y = \ddot{y} = -g \quad a_z = \ddot{z} = 0 \]

  with initial conditions,
  \[ x_0 = y_0 = z_0 = 0 \quad (v_x)_0, (v_y)_0, (v_z)_0 = 0 \]

  Integrating twice yields
  \[ v_x = (v_x)_0 \quad v_y = (v_y)_0 - gt \quad v_z = 0 \]
  \[ x = (v_x)_0 t \quad y = (v_y)_0 t - \frac{1}{2} gt^2 \quad z = 0 \]

- Motion in horizontal direction is uniform.
- Motion in vertical direction is uniformly accelerated.
- Motion of projectile could be replaced by two independent rectilinear motions.
Vector Mechanics for Engineers: Dynamics
Tangential and Normal Components

- Velocity vector of particle is tangent to path of particle. In general, acceleration vector is not. Wish to express acceleration vector in terms of tangential and normal components.

- \( \vec{e}_t \) and \( \vec{e}_n \) are tangential unit vectors for the particle path at \( P \) and \( P' \). When drawn with respect to the same origin, \( \Delta \vec{e}_t = \vec{e}_t' - \vec{e}_t \) and \( \Delta \theta \) is the angle between them.

\[ \Delta e_t = 2 \sin(\Delta \theta/2) \]

\[ \lim_{\Delta \theta \to 0} \frac{\Delta e_t}{\Delta \theta} = \lim_{\Delta \theta \to 0} \frac{\sin(\Delta \theta/2)}{\Delta \theta/2} \quad \vec{e}_n = \vec{e}_n \]

\[ \vec{e}_n = \frac{d\vec{e}_t}{d\theta} \]

With the velocity vector expressed as \( \vec{v} = \vec{v}_t \), the particle acceleration may be written as

\[ \vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt} \vec{e}_t + v \frac{d\vec{e}_t}{dt} = \frac{dv}{dt} \vec{e}_t + v \frac{d\vec{e}_t}{d\theta} \frac{d\theta}{dt} \frac{ds}{d\theta} \]

but

\[ \frac{d\vec{e}_t}{d\theta} = \vec{e}_n \quad \rho \frac{d\theta}{dt} = ds \quad \frac{ds}{dt} = v \]

After substituting,

\[ \vec{a} = \frac{dv}{dt} \vec{e}_t + \frac{v^2}{\rho} \vec{e}_n \quad a_t = \frac{dv}{dt} \quad a_n = \frac{v^2}{\rho} \]

- Tangential component of acceleration reflects change of speed and normal component reflects change of direction.

- Tangential component may be positive or negative. Normal component always points toward center of path curvature.
**Tangential and Normal Components**

- Relations for tangential and normal acceleration also apply for particle moving along space curve.
  \[
  \ddot{a} = \frac{dv}{dt} \dot{e}_t + \frac{v^2}{\rho} \dot{e}_n \\
  a_t = \frac{dv}{dt} \\
  a_n = \frac{v^2}{\rho}
  \]

- Plane containing tangential and normal unit vectors is called the osculating plane.

- Normal to the osculating plane is found from

\[
\dot{e}_b = \dot{e}_t \times \dot{e}_n
\]

\[\dot{e}_n = \text{principal normal}\]

\[\dot{e}_b = \text{binormal}\]

- Acceleration has no component along binormal.

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**Sample Problem 11.10**

A motorist is traveling on curved section of highway at 60 mph. The motorist applies brakes causing a constant deceleration rate.

Knowing that after 8 s the speed has been reduced to 45 mph, determine the acceleration of the automobile immediately after the brakes are applied.

**SOLUTION:**

- Calculate tangential and normal components of acceleration.

- Determine acceleration magnitude and direction with respect to tangent to curve.
**Sample Problem 11.10**

**SOLUTION:**

- Calculate tangential and normal components of acceleration.
  
  \[
  a_t = \frac{\Delta v}{\Delta t} = \frac{(66 - 88) \text{ ft/s}}{8 \text{ s}} = -2.75 \frac{\text{ft}}{\text{s}^2}
  \]
  
  \[
  a_n = \frac{v^2}{\rho} = \frac{(88 \text{ ft/s})^2}{2500 \text{ ft}} = 3.10 \frac{\text{ft}}{\text{s}^2}
  \]

- Determine acceleration magnitude and direction with respect to tangent to curve.
  
  \[
  a = \sqrt{a_t^2 + a_n^2} = \sqrt{(-2.75)^2 + 3.10^2} \quad \Rightarrow \quad a = 4.14 \frac{\text{ft}}{\text{s}^2}
  \]

  \[
  \alpha = \tan^{-1} \frac{a_n}{a_t} = \tan^{-1} \frac{3.10}{2.75} \quad \Rightarrow \quad \alpha = 48.4^\circ
  \]

**Sample Problem 11.11**
Radial and Transverse Components

- When particle position is given in polar coordinates, it is convenient to express velocity and acceleration with components parallel and perpendicular to \(OP\).

- The particle velocity vector is
  \[
  \dot{\mathbf{v}} = \frac{d}{dt}(r \hat{\mathbf{e}}_r) + \frac{dr}{dt} \hat{\mathbf{e}}_r + r \frac{d\hat{\mathbf{e}}_r}{dt} + r \frac{d\hat{\mathbf{e}}_\theta}{dt} + \hat{\mathbf{e}}_\theta
  \]
  \[
  = \dot{r} \hat{\mathbf{e}}_r + r \dot{\theta} \hat{\mathbf{e}}_\theta
  \]

- Similarly, the particle acceleration vector is
  \[
  \ddot{\mathbf{a}} = \frac{d}{dt}\left(\dot{\mathbf{v}}\right) = \frac{d^2}{dt^2}(r \hat{\mathbf{e}}_r) + \frac{dr}{dt} \frac{d\hat{\mathbf{e}}_r}{dt} + r \frac{d^2\hat{\mathbf{e}}_r}{dt^2} + r \frac{d\hat{\mathbf{e}}_\theta}{dt} + \frac{d\dot{\theta}}{dt} \hat{\mathbf{e}}_\theta
  \]
  \[
  = \left(\ddot{r} - r \dot{\theta}^2\right) \hat{\mathbf{e}}_r + \left(r \ddot{\theta} + 2 \dot{r} \dot{\theta}\right) \hat{\mathbf{e}}_\theta
  \]

- When particle position is given in cylindrical coordinates, it is convenient to express the velocity and acceleration vectors using the unit vectors \(\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_\theta, \text{and} \hat{\mathbf{k}}\).

- Position vector,
  \[
  \mathbf{r} = R \hat{\mathbf{e}}_R + z \hat{\mathbf{k}}
  \]

- Velocity vector,
  \[
  \mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{R} \hat{\mathbf{e}}_R + R \dot{\theta} \hat{\mathbf{e}}_\theta + \dot{z} \hat{\mathbf{k}}
  \]

- Acceleration vector,
  \[
  \ddot{\mathbf{a}} = \frac{d\mathbf{v}}{dt} = \left(\ddot{R} - R \dot{\theta}^2\right) \hat{\mathbf{e}}_R + \left(R \ddot{\theta} + 2 \dot{R} \dot{\theta}\right) \hat{\mathbf{e}}_\theta + \ddot{z} \hat{\mathbf{k}}
  \]
Sample Problem 11.12

Rotation of the arm about O is defined by \( \theta = 0.15t^2 \) where \( \theta \) is in radians and \( t \) in seconds. Collar B slides along the arm such that \( r = 0.9 - 0.12t^2 \) where \( r \) is in meters.

After the arm has rotated through 30\(^o\), determine (a) the total velocity of the collar, (b) the total acceleration of the collar, and (c) the relative acceleration of the collar with respect to the arm.

**SOLUTION:**

- Evaluate time \( t \) for \( \theta = 30^o \).
- Evaluate radial and angular positions, and first and second derivatives at time \( t \).
- Calculate velocity and acceleration in cylindrical coordinates.
- Evaluate acceleration with respect to arm.

\[
\begin{align*}
\theta &= 0.15t^2 \\
&= 30^o = 0.524 \text{ rad} \\
t &= 1.869 \text{ s}
\end{align*}
\]

\[
\begin{align*}
r &= 0.9 - 0.12t^2 = 0.481 \text{ m} \\
\dot{r} &= -0.24t = -0.449 \text{ m/s} \\
\ddot{r} &= -0.24 \text{ m/s}^2 \\
\theta &= 0.15t^2 = 0.524 \text{ rad} \\
\dot{\theta} &= 0.30t = 0.561 \text{ rad/s} \\
\ddot{\theta} &= 0.30 \text{ rad/s}^2
\end{align*}
\]
Sample Problem 11.12

- Calculate velocity and acceleration.

\[
v_r = \dot{r} = -0.449 \text{ m/s}
\]

\[
v_\theta = r\ddot{\theta} = (0.481 \text{ m})(0.561 \text{ rad/s}) = 0.270 \text{ m/s}
\]

\[
v = \sqrt{v_r^2 + v_\theta^2}
\]

\[
\beta = \tan^{-1} \frac{v_\theta}{v_r}
\]

\[
v = 0.524 \text{ m/s} \quad \beta = 31.0^\circ
\]

\[
a_r = \ddot{r} - r\dot{\theta}^2
\]

\[
= -0.240 \text{ m/s}^2 - (0.481 \text{ m})(0.561 \text{ rad/s})^2
\]

\[
= -0.391 \text{ m/s}^2
\]

\[
a_\theta = r\dddot{\theta} + 2\dot{r}\dot{\theta}
\]

\[
= (0.481 \text{ m})(0.3 \text{ rad/s}^2) + 2(-0.449 \text{ m/s})(0.561 \text{ rad/s})
\]

\[
= -0.359 \text{ m/s}^2
\]

\[
a = \sqrt{a_r^2 + a_\theta^2}
\]

\[
\gamma = \tan^{-1} \frac{a_\theta}{a_r}
\]

\[
a = 0.531 \text{ m/s} \quad \gamma = 42.6^\circ
\]

Sample Problem 11.12

- Evaluate acceleration with respect to arm.

Motion of collar with respect to arm is rectilinear and defined by coordinate \( r \).

\[
\dot{\mathbf{a}}_{B/OA} = \ddot{r} = -0.240 \text{ m/s}^2
\]
Motion Relative to a Frame in Translation

- Designate one frame as the fixed frame of reference. All other frames not rigidly attached to the fixed reference frame are moving frames of reference.
- Position vectors for particles $A$ and $B$ with respect to the fixed frame of reference $Oxyz$ are $\vec{r}_A$ and $\vec{r}_B$.
- Vector $\vec{r}_{B/A}$ joining $A$ and $B$ defines the position of $B$ with respect to the moving frame $Ax'y'z'$ and $\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$.
- Differentiating twice, $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$ $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ velocity of $B$ relative to $A$.
- $\vec{a}_{B/A}$ acceleration of $B$ relative to $A$.
- Absolute motion of $B$ can be obtained by combining motion of $A$ with relative motion of $B$ with respect to moving reference frame attached to $A$.

Motion of Several Particles: Relative Motion

- For particles moving along the same line, time should be recorded from the same starting instant and displacements should be measured from the same origin in the same direction.
  - $x_{B/A} = x_B - x_A = \text{relative position of } B \text{ with respect to } A$
  - $x_B = x_A + x_{B/A}$

  - $v_{B/A} = v_B - v_A = \text{relative velocity of } B \text{ with respect to } A$
  - $v_B = v_A + v_{B/A}$

  - $a_{B/A} = a_B - a_A = \text{relative acceleration of } B \text{ with respect to } A$
  - $a_B = a_A + a_{B/A}$
Sample Problem 11.4

Ball thrown vertically from 12 m level in elevator shaft with initial velocity of 18 m/s. At same instant, open-platform elevator passes 5 m level moving upward at 2 m/s.

Determine (a) when and where ball hits elevator and (b) relative velocity of ball and elevator at contact.

SOLUTION:

- Substitute initial position and velocity and constant acceleration of ball into general equations for uniformly accelerated rectilinear motion.

\[ v_B = v_0 + at = 18 \frac{m}{s} - \left( 9.81 \frac{m}{s^2} \right) t \]
\[ y_B = y_0 + v_0t + \frac{1}{2} at^2 = 12 \text{ m} + \left( 18 \frac{m}{s} \right) t - \left( 4.905 \frac{m}{s^2} \right) t^2 \]

- Substitute initial position and constant velocity of elevator into equation for uniform rectilinear motion.

\[ v_E = \frac{2 \text{ m}}{s} \]
\[ y_E = y_0 + v_Et = 5 \text{ m} + \left( \frac{2 \text{ m}}{s} \right) t \]
Sample Problem 11.4

- Write equation for relative position of ball with respect to elevator and solve for zero relative position, i.e., impact.

\[
y_{B/E} = \left[ 12 + 18t - 4.905t^2 \right] - (5 + 2t) = 0
\]

\[t = -0.39 \text{ s (meaningless)}\]
\[t = 3.65 \text{ s}\]

- Substitute impact time into equations for position of elevator and relative velocity of ball with respect to elevator.

\[
y_E = 5 + 2(3.65)
\]
\[
v_{B/E} = (18 - 9.81t) - 2
\]
\[
= 16 - 9.81(3.65)
\]
\[
v_{B/E} = -19.81 \text{ m/s}
\]