**EC 203 - INTERMEDIATE MICROECONOMICS**
**Boğaziçi University - Department of Economics**
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**Problem Set 7 - Solutions**

1. Suppose a firm has the production function $f(L, K) = L^{1/3}K^{1/3} - 3$. The prices of the output, labor and capital are $p = 9$, $w = 3$, $r = 1$, respectively.

   (a) If the firm is operating in the short-run and has a fixed capital amount $\overline{K} = 8$, then find the short-run profit maximizing labor amount.

   **Solution:** The profit maximization problem in the short-run is given by
   $max_{q,L} \quad pq - wL - r\overline{K} = 9[L^{1/3}\overline{K}^{1/3} - 3] - 3L - \overline{K} = 9[L^{1/3}8^{1/3} - 3] - 3L - 8 = 9[2L^{1/3} - 3] - 3L - 8.$
   Then the first order condition is $9[2/3(L^{-2/3})] = 3$, that is, $L^{-2/3} = 1/2$ or $L^{2/3} = 2$, which implies $L^*_{SR} = 8^{1/2}$.

   (b) Suppose now the firm operates in the long-run. Find the profit maximizing capital and labor levels.

   **Solution:** The profit maximization problem in the long-run is given by $max_{q,L,K} \quad pq - wL - rK$, which is $max_{q,L,K} \quad 9[L^{1/3}K^{1/3} - 3] - 3L - K$. The first order conditions are as follows:
   For $L$, we get $9[(1/3)L^{-2/3}K^{1/3}] = 3$, that is, $L^{-2/3}K^{1/3} = 1$
   For $K$, we get $9[(1/3)L^{1/3}K^{-2/3}] = 1$, that is, $L^{1/3}K^{-2/3} = 1/3$
   Combining these two we get, $K = 3L$.
   Then, the profit is $9[L^{1/3}(3L)^{1/3} - 3] - 3L - 3L = 9[L^{2/3}(3)^{1/3} - 3] - 6L$.
   The first order condition is $9[(3)^{1/3}(2/3)L^{-1/3}] = 6$, that is, $(3)^{1/3}L^{-1/3} = 1$ that is $L = 3$, and $K = 9$. However the profit is $9[3^{1/3}9^{1/3} - 3] - 9 - 9 = 9[3^{1/3}3 - 3] - 18 < 0$. Thus, the optimal thing to do is to get $L^*_{LR} = 0$, $K^*_{LR} = 0$ and produce zero in the long-run (exit the market), $q^* = 0$.

2. Suppose a firm has the production function $f(L, K) = L + 2K$. The output price is $p$. The prices of capital and labor are $r$ and $w$, respectively.

   (a) Find the cost minimizing $L^*$ and $K^*$ as functions of $r$ and $w$. Suppose $r \neq 2w$.

   **Solution:** Think about the cost minimization problem to produce a certain amount $q$. Because the MRTS is not diminishing we will have corner solution and we compare $MP_L/w$ and $MP_K/r$, where $MP_L = 1$ and $MP_K = 2$.
   If $1/w > 2/r$, then produce all $q$ with $L$, that is if $r/w > 2$, $L = q$ and $K = 0$: $c(q) = wL = wq$.
   If $1/w < 2/r$, then produce all $q$ with $K$, that is if $r/w < 2$, $L = 0$ and $K = q/2$: $c(q) = rK = rq/2$.
   $L = q$ if $2w < r$, $\quad 0$ if $2w > r$, and $K = 0$ if $2w < r$, $\quad q/2$ if $2w > r$

   (b) Assume the prices are $p = 12$, $w = 15$ and $r = 10$. Suppose the firm has a capacity constraint: it can produce at most 100 units. Find the profit maximizing quantity, and input combination. Find the profit level?

   **Solution:** Note that $pMP_L = 12 \cdot 1 = 12 < 15 = w$. And, $pMP_L = 12 \cdot 2 = 24 > 10 = r$. Thus, increase $K$ and decrease $L$. Since $MP_L$ and $MP_K$ are constants (note MRTS is not diminishing), need to decrease $L$ all the way down to 0, and produce all $q$ with $K$, that is $K = q/2$. Then the profit function is $12q - 10K = 12q - 10(q/2) = 12q - 5q = 7q$ which is maximized at $q = \infty$, but there is a capacity constraint of 100 units, thus $q^* = 100$. Thus, $L^* = 0$ and $K^* = 50$. The profit is $\pi = 7q^* = 700$.

(c) Suppose that the firm is required to produce at full capacity $q = 100$. Prices are as in part (b). Show that the cost minimizing input levels are exactly the same levels you found in part (b).

**Solution:** From part (a) we know that $L = 0$ and $K = q/2$ since at prices $w = 15$ and $r = 10$, we have $r/w < 2$. Thus, cost minimizing levels are $L^* = 0$ and $K^* = 50$.

3. Suppose a firm has the production function $f(L, K) = K^{1/4}L^{1/2}$. The output price is $p$. The prices of capital and labor are $r$ and $w$, respectively.

(a) Find the profit maximizing input demands $K^*(p, r, w)$ and $L^*(p, r, w)$.

**Solution:** We can first write profits as a function of the choice variables $K$ and $L$ as follows:

$$\pi(K, L, p, r, w) = pq - wL - rK = pK^{1/4}L^{1/2} - wL - rK.$$

Taking the first order conditions with respect to the choice variables $K$ and $L$, respectively, yields

$$\frac{1}{4}pK^{-3/4}L^{1/2} = r$$

$$\frac{1}{2}pK^{1/4}L^{-1/2} = w$$

Solving the two equations together for the two unknowns $K$ and $L$ yields

$$\begin{aligned} K^*(p, r, w) &= \frac{p^4}{64r^2w^2}, \\ L^*(p, r, w) &= \frac{p^4}{32rw^3}. \end{aligned}$$

(b) Find the profit maximizing output $q^*$ and profit in terms of $p, r$ and $w$.

**Solution:** The output is given simply by

$$q^*(p, r, w) = q(K^*(p, r, w), L^*(p, r, w)) = \left(\frac{p^4}{64r^2w^2}\right)^{1/4}\left(\frac{p^4}{32rw^3}\right)^{1/2} = \frac{p^3}{16rw^2}$$

and the profit function by

$$\begin{aligned} \pi(p, r, w) &= pq(K^*(p, r, w), L^*(p, r, w)) - wL^*(p, r, w) - rK^*(p, r, w) \\ &= p\frac{p^3}{16rw^2} - w\frac{p^4}{32rw^3} - r\frac{p^4}{64r^2w^2} \\ &= \frac{p^4}{16rw^2} - \frac{p^4}{32rw^2} - \frac{p^4}{64rw^2} \\ &= \frac{p^4}{64rw^2} \end{aligned}$$

(c) Law of Output Supply states that the output supply is increasing in output price, and Law of Input Demand states that input demands $K^*(p, r, w)$ and $L^*(p, r, w)$ are decreasing in their own prices, $r$ and $w$ respectively? Are your answers consistent with the Law of Output Supply and Law of Input Demand? Explain.

**Solution:** From our answers in part (a), we can see that $L^*$ is decreasing in $w$ and $K^*$ is decreasing in $r$. The output is increasing in $p$. Thus, both Law of Output Supply and Law of Input Demands hold.

(d) In the short-run, suppose capital is fixed at $\bar{K} = \frac{1}{64}$. Find the resulting short-run input labor demand and output supply functions, $L^{SR}(p, r, w; \bar{K})$ and $q^{SR}(p, r, w; \bar{K})$.

**Solution:** We have that $\bar{K} = \frac{1}{64}$, so the firm is now maximizing the profit function

$$\begin{aligned} \pi(\bar{K}, L, p, r, w) &= pq(\bar{K}, L) - wL - r\bar{K} \\ &= p\bar{K}^{1/4}L^{1/2} - wL - r\bar{K} \\ &= p\left(\frac{1}{64}\right)^{1/4} L^{1/2} - wL - r\left(\frac{1}{64}\right) \end{aligned}$$

with $L$ as its only remaining choice variable. Taking the first order condition with respect to $L$, we get

$$\frac{1}{2}p\left(\frac{1}{64}\right)^{1/4} L^{-1/2} = w$$

and solving for $L$ yields

$$L^{SR}(p, r, w; \bar{K}) = \frac{p^2}{32w^2}$$

The short-run output is then given by

$$q^{SR}(p, r, w; \bar{K}) = q(\bar{K}, L^{SR}(p, r, w; \bar{K})) = \bar{K}^{1/4}L^{SR}(p, r, w; \bar{K})^{1/2} = \left(\frac{1}{64}\right)^{1/4}\left(\frac{p^2}{32w^2}\right)^{1/2} = \frac{p}{16w}.$$

4. Suppose a firm has the production function $f(K, L) = \sqrt{\min\{K, L\}}$. The output price is $p$. The prices of capital and labor are $r$ and $w$, respectively. Find the firm's supply function $q^*(p, r, w)$, by the help of the solution to the cost-minimization problem.

**Solution:** The firm's cost minimization problem is

$$\begin{aligned} \min_{K,L\geq 0} \quad & rK + wL \\ \text{subj. to} \quad & \sqrt{\min\{K, L\}} \geq q, \end{aligned}$$

so writing the production constraint as an equality yields $\min\{K, L\} = q^2$, and since the firm's output depends on the minimum of $K$ and $L$ it will always minimize cost to choose $K = L$, and so we immediately get $K = q^2$ and $L = q^2$, so conditional input demands are

$$K(q, r, w) = q^2 \quad \text{and} \quad L(q, r, w) = q^2$$

and so the cost function becomes

$$\begin{aligned} c(q, r, w) &= rK(q, r, w) + wL(q, r, w) \\ &= (r + w)q^2. \end{aligned}$$

The firm maximizes profit by choosing the output $q$ which solves

$$\max_{q\geq 0} pq - c\left(q, r, w\right),$$

which in this case becomes

$$\max_{q\geq 0} pq - \left(r + w\right) q^2.$$

The FOC yields the $p = MC$ condition

$$p - 2\left(r + w\right) q = 0 \Leftrightarrow q = \frac{p}{2\left(r + w\right)}.$$

and the second derivative is negative, and so the firm's supply function is

$$q^*\left(p, r, w\right) = \frac{p}{2\left(r + w\right)}.$$

5. The cost function for a firm is $c(q) = 10 + 10q + q^2$, where $q$ is output level. Assume the market, that this firm operates in, is perfectly competitive.

   (a) What $q$ should the firm choose to maximize its profit if the market price is $p = 50$?

   **Solution:** In a competitive market a firm will have $p = MC(q)$ which yileds $p = 10 + 2q$, that is, $q = \frac{p-10}{2}$. At $p = 50$, $q = (50 - 10)/2 = 20$. Then the profit is $pq - c(q) = 50 \cdot 20 - [10 + 10 \cdot 20 + 20^2] = 1000 - 10 - 200 - 400 = 390 > 0$. Thus, the firm actually produces $q^* = 20$.

   (b) What is the short-run supply curve of this firm?

   **Solution:** The short-run supply is the marginal cost curve above the average variable cost curve. We have $AVC(q) = 10 + q$ which attains it's minimum at $q = 0$, thus short-run supply is the entire $MC(q) = 10 + 2q$, that is, $q^{SR}$ is given by $q^{SR} = \frac{p-10}{2}$ for all $p \geq 10$, and $q^{SR} = 0$ for all $p < 10$.

6. The cost function for a firm is $c(q) = 10 + 10q - q^2 + (1/3)q^3$, where $q$ is output level. Assume the market, that this firm operates in, is perfectly competitive.

   (a) What is its profit-maximizing condition? What is its short-run supply curve?

   **Solution:** Marginal cost is given by $MC(q) = 10 - 2q + q^2$. Thus the profit maximizing condition is $p = 10 - 2q + q^2$, or $q(q - 2) = p - 10$.

   The short-run supply is given by the $MC(q)$ curve above the $AVC(q)$ curve, where $AVC(q) = 10 - q + (1/3)q^2$. Since MC intersects AVC at its minimum, we can find the intersection point first: $MC(q) = AVC(q)$ that is $10 - 2q + q^2 = 10 - q + (1/3)q^2$, which implies $q^2 = q + (1/3)q^2$, or $q = 1 + (1/3)q$, that is $q = 3/2$. At $q = 3/2$, $MC = AVC = 9.25$. Thus, the SR supply is given by: If $p < 9.25$ choose $q = 0$, and if $p \geq 9.25$, choose $q$ such that $p = 10 - 2q + q^2$.

   (b) What is the exit decision for this firm if $p = 13$?

   **Solution:** Exit if $p < ATC(q)$, that is, $13 < \frac{10}{q} + 10 - q + \frac{1}{3}q^2$, or $3 < \frac{10}{q} - q + \frac{1}{3}q^2$, where $q$ is such that $13 = MC(q) = 10 - 2q + q^2$, that is, $3 = -2q + q^2$, or $q^2 - 2q - 3 = 0$ which is equivalent to $(q - 3)(q + 1) = 0$, thus $q = 3$ ($q = -1$ is not possible). Plug $q = 3$ into $ATC(q) = \frac{10}{q} + 10 - q + \frac{1}{3}q^2 = (10/3) + 10 - 3 + (1/3)9 = 10 + 10/3 = 13.3$ and we get $13 = p < ATC = 13.3$, thus firm exits.

7. The cost function for a firm is

$$c(q) = \begin{cases} 4 & \text{if } q = 0 \\ \\ q^2 + 20 & \text{if } q > 0 \end{cases}$$

(a) Find the lowest price the firm will operate in the short-run, that is the minimum price that the firm will not shut down.

**Solution:** The firm will operate as long as $p \geq AVC(q)$, and when it operates it will choose $q$ such that $p = MC(q)$. Therefore, the firm will operate as long as $p \geq p^{\text{min AVC}}$,which satisfies

$$p^{\text{min AVC}} = MC\left(q^{\text{min AVC}}\right) = AVC\left(q^{\text{min AVC}}\right).$$

We have that

$$MC(q) = c'(q) = 2q$$

while

$$\begin{aligned} AVC(q) &= \frac{c(q) - c(0)}{q} \\ &= \frac{q^2 + 20 - 4}{q} \\ &= \frac{q^2 + 16}{q} \end{aligned}$$

and so

$$\begin{aligned} MC\left(q^{\text{min AVC}}\right) &= AVC\left(q^{\text{min AVC}}\right) \\ \Leftrightarrow 2q^{\text{min AVC}} &= \frac{\left(q^{\text{min AVC}}\right)^2 + 16}{q^{\text{min AVC}}} \\ \Leftrightarrow q^{\text{min AVC}} &= 4, \end{aligned}$$

and so

$$\begin{aligned} p^{\text{min AVC}} &= MC\left(q^{\text{min AVC}}\right) = AVC\left(q^{\text{min AVC}}\right) \\ &= 2(4) = \frac{4^2 + 16}{4} = 8. \end{aligned}$$

The lowest price at which this firm will choose to operate rather than shut down is $p^{\text{min AVC}} = 8$.

(b) Find the short-run supply of this firm.

**Solution:** The firm's supply function is therefore vertical along the vertical axis until $p = 8$, and then jumps to $q = 4$ and follows the graph of $p = 2q$, i.e. $q^*(p) = \frac{p}{2}$, forever beyond that:

$$q^*(p) = \begin{cases} 0 & \text{if } p < 8 \\ \\ \frac{p}{2} & \text{if } p \geq 8. \end{cases}$$