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Asymptotic expansions of multiply scattered surface currents

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We have recently uncovered the convergence characteristics of multiple scattering iterations for “two-dimensional” as well as “three-dimensional scalar (acoustics)” scattering models in the high-frequency regime. As we have demonstrated, a most distinctive property of these later models, compared to their two-dimensional counterparts, is the dependence of corresponding asymptotic expansions on the *relative angle of rotation* between the principal axes of the successive reflection points of the optical rays. Concerning the case of fully “three-dimensional vector (electromagnetic)” scattering problems, here we show that the vectorial nature of the problem, in turn, gives rise to new additional complex structure.

1 Introduction

We have recently uncovered the convergence characteristics of multiple scattering iterations for “two-dimensional” [3] as well as “three-dimensional scalar (acoustics)” [1] scattering models in the high-frequency regime. As we have demonstrated, a most distinctive property of these later models, compared to their two-dimensional counterparts, is the dependence of corresponding asymptotic expansions on the *relative angle of rotation* between the principal axes of the successive reflection points of the optical rays. Concerning the case of fully “three-dimensional vector (electromagnetic) scattering” problems, here we show that their vectorial nature in turn, gives rise to new additional complex structure. More precisely, we show that at each reflection the *asymptotic* currents are, as they ought to be, tangential to the surfaces and, most importantly, they undergo a rotation and a projection onto the surface perpendicular to the reflection vector, followed by a second rotation and a projection onto the tangent space at the point of arrival.

2 Integral Equations and Multiple Scattering Formulation

We consider the scattering of a time-harmonic electromagnetic plane-wave $(\mathbf{E}^{inc}, \mathbf{H}^{inc}) = e^{ik\alpha\cdot x}(\mathbf{A}\times\alpha, \mathbf{A})$ in $\mathbb{R}^3$, $|\alpha| = 1$, $\mathbf{A}\cdot\alpha = \mathbf{0}$, from a perfectly conducting obstacle K in $\mathbb{R}^3$ whose boundary ∂K is a 2-dimensional regular compact manifold. The classical frequency-domain electromagnetic scattering problem then, consists of determining the scattered electromagnetic field $(\mathbf{E}, \mathbf{H})$ satisfying the equations

$$[\nabla\times(\mathbf{E},\mathbf{H}) = ik(\mathbf{H},\mathbf{E}) \quad\text{and}\quad \nabla\cdot(\mathbf{E},\mathbf{H}) = 0] \text{ in } \mathbb{R}^3\backslash\overline{K}, \quad\text{and}\quad \big[\nu\times\mathbf{E} = -\nu\times\mathbf{E}^{inc}\big] \text{ on } \partial K, \tag{1}$$

as well as the Silver-Müller radiation conditions at infinity. As is well known, the scattered electromagnetic field $(\mathbf{E}, \mathbf{H})$ can then be recovered through a knowledge of the surface current $\mathbf{J} = \nu\times(\mathbf{H}+\mathbf{H}^{inc})$ via the classical Stratton-Chu formulae [2]. Although a variety of integral equations for the unknown surface current $\mathbf{J}$ exists, here we choose to utilize an integral equation of the second kind, the magnetic field integral equation (MFIE)

$$\mathbf{J}-\mathcal{R}\mathbf{J} = 2\,\nu\times\mathbf{H}^{inc}\text{ on }\partial K \quad\text{where}\quad (\mathcal{R}\mathbf{J})(x) = \nu(x)\times\int_{\partial K}\nabla_y G(x,y)\times\mathbf{J}(y)dS(y), \quad G(x,y) = -\frac{e^{ik|x-y|}}{2\pi|x-y|}, \tag{2}$$

as this type of operator equations allow a rather straightforward multiple scattering reformulation on domains with many connected components. In fact, when the obstacle K is the disjoint union of finitely many sub-scaterrers $K = \bigcup\{K_\sigma : \sigma\in\mathcal{I},\ |\mathcal{I}| < \infty\}$, writing the MFIE in coordinate form, and multiplying this equation with the inverse of the diagonal operator $(\mathcal{D})_{\sigma_j,\sigma_\ell} = (I_{\sigma_j} - \mathcal{R}_{\sigma_j\sigma_j})\delta_{\sigma_j,\sigma_\ell}$ yields the alternative operator equation of the second kind

$$(I-\mathcal{T})\mathbf{J} = \mathcal{D}^{-1}\mathbf{F} \quad\text{where}\quad \mathcal{T}_{\sigma\tau} = [(I_\sigma - \mathcal{R}_{\sigma\sigma})^{-1}\mathcal{R}_{\sigma\tau}]\delta_{\sigma,\tau} \quad\text{and}\quad \mathbf{F}_\sigma = 2\,\nu\times\mathbf{H}^{inc}\big|_{\partial K_\sigma}\,; \tag{3}$$

accordingly, considering now the Neumann series solution to (3), we see that the surface current $\mathbf{J}$ is the superposition over all *primitive periodic orbits* $\{\sigma_m\}_{m\geq 0}\in\mathcal{I}^{\mathbb{N}}$ (cf. [3]) of the *iterated surface currents* $\mathbf{J}_{\sigma_m}$ that are defined as the solutions of

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the following recursion:

$$\mathbf{J}_{\sigma_m}(x)-\nu(x)\times\int_{\partial K_{\sigma_m}}\nabla_y G(x,y)\times\mathbf{J}_{\sigma_m}(y)\,dS(y)=\nu(x)\times\begin{cases}2\,\mathbf{H}^{inc}(x)\big|_{\partial K_{\sigma_0}} & m=0,\\[2ex] \displaystyle\int_{\partial K_{\sigma_{m-1}}}\nabla_y G(x,y)\times\mathbf{J}_{\sigma_{m-1}}(y)\,dS(y), & m\geq 1.\end{cases}\tag{4}$$

3 Asymptotic Expansions of Iterated Surface Currents

We now suppose that the obstacles $\{K_\sigma : \sigma\in\mathcal{I}\}$ are *convex* and satisfy the *visibility* and *no-occlusion* conditions (cf. [1, 3]), and let $\{\sigma_m\}_{m\geq 0}\in\mathcal{I}^\infty$ be a multiple-scattering sequence (not necessarily periodic) in the sense that $\sigma_{m+1}\neq\sigma_m$ for all $m\geq 0$. In what follows, we will write K_m, $\mathbf{J}_m$, … instead of K_{σ_m}, $\mathbf{J}_{\sigma_m}$ etc. For $x\in\partial K_m$, we denote by $(x_0^m(x),\ldots,x_m^m(x))\in\partial K_0\times\cdots\times\partial K_m$ the "broken $(m+1)$-ray terminating at x" (cf. [1, 3]). Finally, denoting by $\Xi_p(x)$ and $\kappa_p(x)$ $(p=1,2)$ the unit vectors directed in principal directions and the principal curvatures, respectively, at $x\in\partial K$, and by $\nu(x)$ the unit normal vector to the surface ∂K at $x\in\partial K$ directed into the exterior region $\mathbb{R}^3\backslash\overline{K}$, our main result concerning the high-frequency asymptotic expansions of iterated surface currents $\mathbf{J}_m$ reads as follows:

Theorem 3.1 (Asymptotic Expansions of Iterated Surface Currents) On any compact subset of the m–th illuminated region (cf. [1, 3]), the iterated surface current $\mathbf{J}_m$ satisfies as $k\to\infty$

$$\mathbf{J}_m(x)=\left(1+\mathcal{O}\left(k^{-1}\right)\right)\begin{cases}2\,\nu(x)\times\mathbf{H}^{inc}, & \textit{if } m=0,\\[1ex] \dfrac{e^{ik|x-x_{m-1}^m(x)|}}{\sqrt{\det N_{m,m}(x)}}\,\nu(x)\times\left(\dfrac{x_{m-1}^m(x)-x}{|x_{m-1}^m(x)-x|}\times\mathbf{J}_{m-1}(x_{m-1}^m(x))\right), & \textit{if } m\geq 1.\end{cases}$$

Here the matrices $N_{m,j}(x)\in\mathbb{R}^2\times\mathbb{R}^2$ are defined recursively as

$$N_{m,1}(x)=2\,\left[x_1^m(x)-x_0^m(x)\right]\cdot\nu(x_0^m(x))\,\kappa_{m,0}(x)+U_{m,0}(x)$$

and, for $j=2,\ldots,m$,

$$\begin{aligned}N_{m,j}(x)&=2\,\left[x_j^m(x)-x_{j-1}^m(x)\right]\cdot\nu(x_{j-1}^m(x))\,\kappa_{m,j-1}(x)+U_{m,j-1}(x)\\&\quad+\frac{|x_j^m(x)-x_{j-1}^m(x)|}{|x_{j-1}^m(x)-x_{j-2}^m(x)|}\left(U_{m,j-1}(x)-T_{m,j-2}(x)\left[N_{m,j-1}(x)\right]^{-1}T_{m,j-2}(x)^t\right)\end{aligned}$$

where $T_{m,j}(x), U_{m,j}(x), \kappa_{m,j}(x)\in\mathbb{R}^{2\times 2}$ are given, for $x\in\partial K_m$ and $j=0,\ldots,m-1$, as

$$T_{m,j}(x)_{pq}=\Xi_p(x_{j+1}^m(x))\cdot\Xi_q(x_j^m(x))-\frac{x_{j+1}^m(x)-x_j^m(x)}{|x_{j+1}^m(x)-x_j^m(x)|}\cdot\Xi_p(x_{j+1}^m(x))\,\frac{x_{j+1}^m(x)-x_j^m(x)}{|x_{j+1}^m(x)-x_j^m(x)|}\cdot\Xi_q(x_j^m(x))$$

$$U_{m,j}(x)_{pq}=\Xi_p(x_j^m(x))\cdot\Xi_q(x_j^m(x))-\frac{x_{j+1}^m(x)-x_j^m(x)}{|x_{j+1}^m(x)-x_j^m(x)|}\cdot\Xi_p(x_j^m(x))\,\frac{x_{j+1}^m(x)-x_j^m(x)}{|x_{j+1}^m(x)-x_j^m(x)|}\cdot\Xi_q(x_j^m(x))$$

$$\kappa_{m,j}(x)_{pq}=\kappa_p(x_j^m(x))\,\delta_{pq}$$

with δ_{pq} denoting the Kronecker *delta.*

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