A TALK ON ‘SMALL’ MULTIPLICATIVE SUBGROUPS OF FIELDS

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1. Statements of the main results

We first state the main theorems without defining the Mann property. However, one can think a multiplicative subgroup of a field with the Mann property as ‘small’ enough so that the addition does not leave a trace on the group. For instance, all the subgroups of \( \mathbb{C}^\times \) of finite rank have the Mann property.

From now on \( K \) is a field, and \( G \) is a subgroup of \( K^\times \). Today I will mostly concentrate the algebraically closed case.

**Proposition 1.1.** For algebraically closed \( K \), the following are equivalent:

1. \( G \) has the Mann Property;
2. for every algebraic set \( V \subseteq K^n \) its trace \( V \cap G^n \) is a finite union of cosets of subgroups of \( G^n \);
3. for every \( X \subseteq K^n \) that is definable in \((K,G)\) its trace \( X \cap G^n \) is definable in the group \( G \).

Let \( \Gamma \) be a subgroup of \( \mathbb{C}^\times \) with the Mann property.

**Theorem 1.2.** Let \( K \) be an algebraically closed field of characteristic 0, let \( G \) be a subgroup of \( K^\times \), and let a map \( \gamma \mapsto \gamma' : \Gamma \to G \) be given. Then \((K,G,(\gamma')_{\gamma \in \Gamma}) \equiv (\mathbb{C},\Gamma,(\gamma)_{\gamma \in \Gamma})\) if and only if

- \((G,(\gamma')_{\gamma \in \Gamma}) \equiv (\Gamma,(\gamma)_{\gamma \in \Gamma})\) as groups with distinguished elements;
- \((K,G,(\gamma')_{\gamma \in \Gamma})\) satisfies the Mann axioms of \( \Gamma \).

I will explain ‘Mann property’, and ‘Mann axioms for \( \Gamma \)’ in a bit.

2. Smallness

Below, \( \mathcal{L} \) is a language, \( \mathcal{M} = (M, \ldots) \) is an \( \mathcal{L} \)-structure, and \( G \subseteq M \). If \( f(G^m) = M \) for some \( m, n \) and some \( f : M^m \to M \) definable in \( \mathcal{M} \), we call \( G \) large in \( \mathcal{M} \), and otherwise we call \( G \) small in \( \mathcal{M} \). In particular, if \( M \) is infinite and \( |G| < |M| \), then \( G \) is small in \( \mathcal{M} \). (It came to our attention that Casanovas and Ziegler have another notion of small. It is easy to see that their notion is equivalent to ours for strongly minimal \( \mathcal{M} \).

We can now state a generalization of a theorem of Keisler on pairs of algebraically closed fields:
**Proposition 2.1.** Let $T$ be an $L$-theory with QE whose models are infinite and strongly minimal. Let $\mathcal{M}$ and $\mathcal{N}$ be models of $T$ with substructures $G = (G, \ldots)$ and $H = (H, \ldots)$, such that

(i) $G$ is small in $\mathcal{M}$ and $H$ is small in $\mathcal{N}$;

(ii) $G \equiv H$ and $\mathcal{M} \equiv N$.

Then $(\mathcal{M}, G) \equiv (\mathcal{N}, H)$.

**Remark.** Suppose $F$ is a subfield of the algebraically closed field $K$. Then $F$ is large in $K$ if and only if $F = K$, or $F$ is a real closed field and $[K : F] = 2$. This follows from a theorem of E. Artin.

3. **Algebraically closed fields with a multiplicative set**

From now on $L = \{0, 1, +, -, \cdot\}$ is the language of rings. We shall also use its sublanguage $L_m := \{1, \cdot\}$ of multiplicative monoids.

Throughout this section $K$ is an algebraically closed field, with prime field $\mathbb{F}$, and $G$ is a multiplicative set in $K$, that is, $G$ is a subset of $K$ that contains 1 and is closed under multiplication. (For example, any subring of $K$ is a multiplicative set in $K$.) We consider $G$ as an $L_m$-structure in the obvious way. The addition of $K$ also leaves a trace on $G$, and to deal with that we extend $L_m$ to the language

$L_m(\Sigma) := \{1, \cdot\} \cup \{\Sigma_k : k \in \mathbb{Z}^n, n = 1, 2, \ldots\}$

of multiplicative monoids with additive relations: here $\Sigma_k$ is an $n$-ary relation symbol for $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. We expand the monoid $G$ to an $L_m(\Sigma)$-structure $G(\Sigma)$ by interpreting $\Sigma_k$ as the $n$-ary relation

$\{(g_1, \ldots, g_n) \in G^n : k_1g_1 + \cdots + k_ng_n = 0\}$
on $G$. As a special case of Proposition 2.1 we have:

**Corollary 3.1.** If $G$ is small in $K$, then $\text{Th}(K, G)$ is completely determined by $\text{Th}(G(\Sigma))$.

In other words, if $G$ is small in $K$, and $H$ is a small multiplicative set in an algebraically closed field $L$, then

$(K, G) \equiv (L, H) \iff G(\Sigma) \equiv H(\Sigma)$.

To see how this follows from 2.1, note that algebraically closed fields can be construed as $L_m(\Sigma)$-structures, that their theory in this language admits QE, and that the substructures of an algebraically closed field $L$ viewed as $L_m(\Sigma)$-structure are exactly the structures $H(\Sigma)$ where $H$ is a multiplicative set in $L$. Also, the characteristic of an algebraically closed field is determined by the sentences $\Sigma_k(1)$ that it satisfies where $1 \leq k \in \mathbb{Z} = \mathbb{Z}^1$.

In the next result $G$ is not assumed to be small in $K$.

**Proposition 3.2.** Every subset of $G^n$ definable in $(K, G)$ is definable in the structure $G(\Sigma)$. 

Remarks.

(1) The proposition deals with a special case of the situation considered in Proposition 3.1 of Pillay in Bouscaren volume, but is a bit stronger in that special case.

(2) In case $G$ is a subring of $K$, Proposition 3.2 says that the subsets of $G^n$ definable in $(K, G)$ are definable in the ring $G$.

(3) The proposition fails badly when $K$ is replaced by the field $\mathbb{R}$. For example, every subset of $\mathbb{Z}$ is definable in $(\mathbb{R}, \mathbb{Z})$.

The following are consequences of a back-and-forth argument:

**Corollary 3.3.** Suppose $G$ is small in $K$ and the $\mathcal{L}_m(\Sigma)$-structure $G(\Sigma)$ is $\omega$-stable. Then the $\mathcal{L}(U)$-structure $(K, G)$ is $\omega$-stable.

In a similar way we obtain:

**Corollary 3.4.** If $G$ is small in $K$ and $G(\Sigma)$ is superstable (stable), then $(K, G)$ is superstable (respectively, stable).

**Corollary 3.5.** Let $G$ be small in $K$, and let $G'$ be an algebraically closed subfield of $K$ with a multiplicative subset $G'$ that is small in $K'$. Suppose $G' \subseteq G$ with $G'(\Sigma) \preceq G(\Sigma)$, and $K'$ and $\mathbb{F}(G')$ are free over $\mathbb{F}(G')$ in $K$. Then $(K', G') \preceq (K, G)$.

4. The Mann Property

Throughout this section, $K$ is a field, $E$ is a subfield of $K$, and $G$ is a subgroup of the multiplicative group $K^\times$ of $K$. Consider a linear equation

\[(*) \quad a_0 = a_1 x_1 + \cdots + a_n x_n,\]

where $a_0, a_1, \ldots, a_n \in K$. A solution of (*) in $G$ is a tuple $(g_1, \ldots, g_n) \in G^n$ such that $a_0 = a_1 g_1 + \cdots + a_n g_n$, and such a solution is said to be nondegenerate if $\sum_{i \in I} a_i g_i \neq 0$ for each non-empty proper subset $I$ of $\{1, \ldots, n\}$.

In the homogeneous case $a_0 = 0$ the set of solutions in $G$ and the set of nondegenerate solutions in $G$ are both unions of orbits with respect to the action of $G$ on $G^n$ defined by $g(g_1, \ldots, g_n) = (gg_1, \ldots, gg_n)$.

Let us say that $G$ has the Mann property over $E$ if each equation

\[a_1 x_1 + \cdots + a_n x_n = 1 \quad (n \geq 2, \ a_1, \ldots, a_n \in E^\times)\]

has only finitely many nondegenerate solutions in $G$. In the case $E = \mathbb{F}$ we just say that $G$ has the Mann property. This terminology is temporary, since the Mann property will be shown to be equivalent to the Mann property over $E$. Consider a homogeneous equation

\[a_1 x_1 + \cdots + a_n x_n = 0, \quad (a_1, \ldots, a_n \in E^\times, n \geq 1).\]

Let $S \subseteq G^n$ be its set of solutions in $G$ and $S_{\text{nd}}$ its subset of nondegenerate solutions in $G$. Then

\[S_{\text{nd}} = \bigcup_{(g_1, \ldots, g_{n-1}) \in S'} (g_1, \ldots, g_{n-1}, 1) G' \subseteq S,\]
where $S'$ is the set of nondegenerate solutions of the equation

$$a_1 y_1 + \cdots + a_{n-1} y_{n-1} = -a_n$$

in $G$. If $G$ has the Mann property over $E$, then $S'$ is finite, so $S_{nd} \subseteq G^n$ is then defined in $G$ by the positive quantifier-free formula

$$\bigvee_{(g_1, \ldots, g_{n-1}) \in S'} x_1 = g_1 x_n \land \cdots \land x_{n-1} = g_{n-1} x_n$$

in the language of $G$-sets. Here we view $G$ as a $G$-set by the (left) action $(g, h) \mapsto gh$. Considering also degenerate solutions of the above homogeneous equation, and using the description above inductively, we obtain:

**Corollary 4.1.** If $G$ has the Mann property over $E$, then $S \subseteq G^n$ is defined in $G$ by the positive quantifier-free formula in the language of $G$-sets. Here we view $G$ as a $G$-set by the (left) action $(g, h) \mapsto gh$. Considering also degenerate solutions of the above homogeneous equation, and using the description above inductively, we obtain:

**Corollary 4.2.** If $K$ is algebraically closed, $G$ has the Mann property and $X \subseteq K^n$ is definable in $(K, G)$, then $X \cap G^n$ is definable in the group $G$.

Having the Mann property is not dependent on the field that $G$ lives in:

**Proposition 4.3.** Suppose that $G$ has the Mann property. Then $G$ has the Mann property over $K$.

**Mann implies Mordell-Lang.** In this subsection $K$ is algebraically closed. For any $n$-tuple $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, consider the character

$$\chi_k : (K^\times)^n \to K^\times, \quad \chi_k(x_1, \ldots, x_n) := x_1^{k_1} \cdots x_n^{k_n}.$$  

This is a multiplicative group homomorphism. For any $d \in \mathbb{N}$, let $D(n, d)$ be the finite collection of subgroups of $(K^\times)^n$ that are intersections of kernels of characters $\chi_k$ with $|k| = |k_1| + \cdots + |k_n| \leq d$. The following yields (1) $\Rightarrow$ (2) of Proposition 1.1.

**Proposition 4.4.** Let $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$ have degree $\leq d$, and let

$$V = \{x \in K^n : f_1(x) = \cdots = f_m(x) = 0\}.$$  

Suppose $G$ has the Mann property. Then $V \cap G^n$ is a finite union of cosets of subgroups $D \cap G^n$ of $G^n$ with $D \in D(n, d)$.

**Proof.** The intersection of finitely many cosets of such subgroups is either empty or again a coset of such a subgroup. Hence we may (and shall) assume that $m = 1$. Put $f := f_1$, and write $f = \sum_{i \in I} a_i X^i$ where all $a_i \in K$ and $I$ is the set of multi-indices $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$ with $|i| = i_1 + \cdots + i_n \leq d$. By Corollary 4.1 the set

$$\{y \in G^{|I|} : \sum_{i \in I} a_i y_i = 0\}$$
is a finite union of finite intersections of subsets of $G^{\left| I \right|}$ of the form
\[
\{y \in G^{\left| I \right|} : gy_i = y_j\}
\]
with $g \in G$ and $i, j \in I$. It remains to observe that for such $g, i, j$ the set
\[
\{x \in G^n : g\chi_i(x) = \chi_j(x)\}
\]
is a coset of the subgroup $D \cap G^n$ of $G^n$ where $D$ is the kernel of $\chi_{i-j}$. □

The proof of the fact that Mordell-Lang implies Mann uses some algebraic group theory, hence I will just state it without a proof.

**Proposition 4.5.** Suppose that for all $a_1, \ldots, a_n \in \mathbb{F}^\times$ the set of solutions of $a_1x_1 + \cdots + a_nx_n = 1$ in $G$ is a boolean combination of subgroups of $G^n$. Then $G$ has the Mann property.

The following lemma is quite technical, however it presents the idea in the core of the Mann property.

**Lemma 4.6.** Let $\Gamma$ be a subgroup of $G$ such that for all $a_1, \ldots, a_n \in \mathbb{E}^\times$ the equation $a_1x_1 + \cdots + a_nx_n = 1$ has the same nondegenerate solutions in $\Gamma$ as in $G$. Then we have for any $g, g_1, \ldots, g_n \in G$:

1. if $g$ is algebraic over $E(\Gamma)$ of degree $d$, then $g^d \in \Gamma$;
2. if $g_1, \ldots, g_n$ are algebraically dependent over $E(\Gamma)$, then they are multiplicatively dependent over $\Gamma$.

5. **Algebraically closed fields with a multiplicative group having the Mann Property**

In this section $K$ is an algebraically closed field and $G$ is a subgroup of $K^\times$ with the Mann property.

**Smallness.** In order to use the results in Section 3 we have to show that $G$ is small in $K$. We shall derive this from the fact that no infinite field is interpretable in any abelian group. This fact follows from three other results:

1. Each abelian group is one-based.
2. If a group is interpretable in a one-based structure, then it has an abelian subgroup of finite index.
3. If $E$ is an infinite field, then the group $\text{SL}_2(E)$ does not have an abelian subgroup of finite index.

**Lemma 5.1.** $G$ is small in $K$.

**Proof.** Suppose towards a contradiction that $G$ is large in $K$. Then we have $K = f(G^m)$ where $f : K^m \rightarrow K$ is definable in $K$. Consider the equivalence relation $E$ on $K^m$ defined by
\[
aEb \iff f(a) = f(b).
\]
By Proposition 3.2 the equivalence relation $E_G := E \cap G^{2m}$ on $G^m$ is definable in the group $G$. The restriction $f|G^m : G^m \to K$ induces a bijection $G^m/E_G \to K$. Using again Proposition 3.2, one checks easily that the addition and multiplication of $K$ correspond under this bijection to binary operations on $G^m/E_G$ that are definable in the many-sorted structure $G^{ni}$ where $G$ is considered as a group. This contradicts the fact that no infinite field is interpretable in any abelian group.

In combination with Lemmas 3.3, 3.4 and Corollary 4.1 this yields:

**Corollary 5.2.** The $L(U)$-structure $(K, G)$ is stable. If the group $G$ is $\omega$-stable (superstable), then $(K, G)$ is $\omega$-stable (respectively, superstable).

Here is another consequence (not used later) of the non-interpretability of infinite fields in abelian groups. Put

$G_0 := G \cup \{0\} \subseteq K,$

$G^{+n} := G_0 + \cdots + G_0 \subseteq K$ (with $n$ summands).

Note that $G_0 = G^{+1} \subseteq G^{+2} \subseteq G^{+3} \subseteq \ldots$.

**Corollary 5.3.** If $G$ is infinite, then the increasing sequence of sets $(G^{+n})$ is strictly increasing.

As an application, for $G = 2\mathbb{Z}\mathbb{Z} \cup -2\mathbb{Z}\mathbb{Z}$, there is no $n$ such that all rational numbers are of the form $a/b$ with $a, b \in G^{+n}$; we do not know how to prove this particular fact other than by the argument above.

**More on the $\omega$-stable case.** Let $H$ be an infinite subgroup of the multiplicative group of a field. Then $H$ has only finitely many elements of any given finite order, so by a theorem of Macintyre, the group $H$ is $\omega$-stable if and only if there is an infinite divisible subgroup $D$ of $H$ and a finite subgroup $B$ of $H$ such that $H = DB$ and $D \cap B = \{1\}$. For such $D$ and $B$ we have $H^{[d]} = D$ for each positive integer $d$ that is a multiple of $|B|$, so the subgroup $D$ does not depend on the particular product decomposition chosen; hence $D$ is a definable subgroup of the group $H$. By Szmielew’s quantifier simplification for abelian groups, $D$ has no infinite proper subgroups definable in the group $H$. Thus, for $\omega$-stable $H$, the subgroup $D$ is the connected component of the group $H$, and $\text{MR}(H^n) = n$ for each $n$, where the Morley rank is with respect to the theory of the group $H$. We use these observations for $H := G$ to prove the following:

**Theorem 5.4.** Suppose $G$ is infinite and $\omega$-stable. Then $\text{MR}(K) = \omega$, where the Morley rank is with respect to the $\omega$-stable theory $\text{Th}(K, G)$.

**Definable sets.** With a mild assumption on $G$ the definable relations in $(K, G)$ are boolean combinations of existentially definable relations. To formulate this precisely, recall that $G^{[d]}$ denotes the subgroup of $d$th powers in $G$. Let $\mathcal{L}(K)$ be the language of rings augmented by names for the elements of $K$, and let $x = (x_1, \ldots, x_m)$ be a tuple of distinct variables.
**Proposition 5.5.** Suppose $G/G[d]$ is finite for each integer $d > 0$. Then every subset of $K^m$ definable in $(K, G)$ is a boolean combination of subsets of $K^m$ defined by formulas $\exists y(U(y) \land \phi(x, y))$ where $\phi(x, y)$ is a quantifier-free $L(K)$-formula.

**Elementary classification within the $\Gamma$-family.** Let $E$ be a field and $\Gamma$ a subgroup of $E^\times$ with the Mann property. All multiplicative groups that contain $\Gamma$ and satisfy the Mann axioms of $\Gamma$ are treated below as members of the same family, the $\Gamma$-family. For example, if $\Gamma = \{1\} \subseteq \mathbb{Q}^\times$, each subgroup of $\mathbb{C}^\times$ generated by algebraically independent elements belongs to the $\Gamma$-family. Likewise, with $E = \mathbb{Q}$ and $\Gamma = 2\mathbb{Z}$, the subgroups $2\mathbb{Z}$ and $2\mathbb{Q}$ of $\mathbb{C}^\times$ belong to the $\Gamma$-family. The formal setting is as follows.

Let $L(U, \Gamma)$ be the language of rings augmented by a unary relation symbol $U$, and by a name (constant symbol) $\gamma$ for each element $\gamma \in \Gamma$. Let $ACF(\Gamma)$ be the theory in the language $L(U, \Gamma)$ whose models are the structures $(K, G, (\gamma')_{\gamma \in \Gamma})$ such that

1. $K$ is an algebraically closed field of the same characteristic as $E$,
2. $G$ is a subgroup of $K^\times$,
3. $\gamma \mapsto \gamma' : \Gamma \rightarrow G$ is a group homomorphism,
4. $(K, G, (\gamma')_{\gamma \in \Gamma})$ satisfies the Mann axioms of $\Gamma$.

Here $\gamma'$ is the interpretation of (the name of) $\gamma$ in $(K, G, (\gamma')_{\gamma \in \Gamma})$. If $E$ is algebraically closed, then $(E, \Gamma, (\gamma)_{\gamma \in \Gamma})$ is clearly a model of $ACF(\Gamma)$. The theory $ACF(\Gamma)$ is never complete, but we can classify its models up to elementary equivalence.

**Theorem 5.6.** Let $(K, G, (\gamma))$ and $(K', G', (\gamma))$ be models of $ACF(\Gamma)$. Then $(K, G, (\gamma)) \equiv (K', G', (\gamma))$ if and only if $(G, (\gamma)) \equiv (G', (\gamma))$ as groups with distinguished elements.