Algebra II

Homework 1 - Due February, 27, 2024

1. Let $f(x) = x^3 + x + 1 \in \mathbb{F}_2[x]$.
   (a) Show that f is irreducible over $\mathbb{F}_2$.
   (b) Let α be a root of f in some extension of $\mathbb{F}_2$. Write the multiplicative inverse of $1 + \alpha$ of the form $a + b\alpha + c\alpha^2$ where $a, b, c \in \mathbb{F}_2$.

2. Let T be an indeterminate and let $K = \mathbb{Q}(T)$ be the field of rational functions in T with rational coefficients. Let $f(x) = x^2 - T \in K[x]$.
   (a) Show that f is irreducible over K.
   (b) Let α be a root of f in some extension of K. Write the multiplicative inverse of $1 + \alpha$ of the form $a + b\alpha$ where $a, b \in K$.

3. Let $E = \mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$ and let $\alpha = \frac{\sqrt[3]{2}+3\sqrt{2}}{-\sqrt[3]{4}+7}$.
   (a) Find the minimal polynomial of α over $\mathbb{Q}$.
   (b) What is $[E : \mathbb{Q}(\alpha)]$?
   (c) Find a basis of E over $\mathbb{Q}$. (You need to prove that the candidate you have found is indeed a basis.)

4. Let T be an indeterminate (over $\mathbb{R}$). Calculate $[\mathbb{R}(T) : \mathbb{R}(T + \frac{1}{T})]$.

5. Let $E|k$ and $F|k$ be two extensions where E, F are contained in a common field. Show that $[EF : k] \leq [E : k][F : k]$. When does the equality hold?

6. Let $E|F$ be a field extension and let $\alpha \in E$ be algebraic over F of odd degree. Show that $F(\alpha) = F(\alpha^2)$.

7. Let $K|\mathbb{Q}$ be an extension of degree 2. Show that $K = \mathbb{Q}(\sqrt{d})$ for some square-free integer $d \neq 1$.