DEFINABLE SETS IN MANN PAIRS

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Abstract. Consider structures $\langle \Omega, k, \Gamma \rangle$ where $\Omega$ is an algebraically closed field of characteristic zero, $k$ is a subfield, and $\Gamma$ is a subgroup of the multiplicative group of $\Omega$. Certain pairs $(k, \Gamma)$ have been singled out as Mann pairs in [3]. We give new examples of such Mann pairs, and for a Mann pair $(k, \Gamma)$ we axiomatize the first-order theory of $\langle \Omega, k, \Gamma \rangle$ in a cleaner way than in [3], and, as the main result of the paper, we characterize the subsets of $\Omega^n$ that are definable in $\langle \Omega, k, \Gamma \rangle$.

1. Introduction

This paper is a sequel to [3]. We let $\Omega$ be an algebraically closed (ambient) field, $k$ a subfield of $\Omega$, and $\Gamma$ a subgroup of $\Omega \times$. Also $m, n$ range over $\mathbb{N} = \{0, 1, 2, \ldots\}$ and for $a \in \Omega$ and $\vec{s} = (s_1, \ldots, s_n) \in \Omega^n$ we put $a\vec{s} := (as_1, \ldots, as_n) \in \Omega^n$. For $n \geq 1$ we set

$$\Gamma^{[n]} := \{ \gamma^n : \gamma \in \Gamma\}$$

(a subgroup of $\Gamma$).

Other notations are explained as needed.

Let $n \geq 2$ and $a_1, \ldots, a_n \in \Omega$. A nondegenerate solution of the equation

$$a_1x_1 + \cdots + a_nx_n = 0$$

is a tuple $\vec{s} = (s_1, \ldots, s_n) \in (\Omega^\times)^n$ such that $a_1s_1 + \cdots + a_ns_n = 0$ and $\sum_{i \in I} a_is_i \neq 0$ for all proper nonempty subsets $I$ of $\{1, \ldots, n\}$; note that then $a_1, \ldots, a_n \neq 0$, and that $a\vec{s}$ for $a \in \Omega^\times$ is also a nondegenerate solution of the same equation, and so for most purposes we can normalize to $s_n = 1$.

Let $n \geq 2$ and define $\Gamma(k, n)$ to be the set of all $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \Gamma^n$ such that $\gamma_n = 1$ and $\vec{\gamma}$ is a nondegenerate solution of some equation

$$a_1x_1 + \cdots + a_nx_n = 0$$

with $a_1, \ldots, a_n \in k$. Recall from [3] that $(k, \Gamma)$ is a Mann pair if and only if $\Gamma(k, n)$ is finite for all $n \geq 2$.\(^1\) One example of a Mann pair is $(\mathbb{Q}, U)$ within the ambient field $\mathbb{C}$, where $U$ is the subgroup of $\mathbb{C}^\times$ consisting of all the roots of unity (see [7] for a proof of this fact). Also, if $\Omega$ is of characteristic zero, $k$ is algebraically closed, $k^\times \cap \Gamma = \{1\}$ and $\Gamma$ is of finite rank (that is, $\Gamma$ has a finitely generated subgroup $\Gamma_0$ such that $\Gamma/\Gamma_0$ is a torsion group), then $(k, \Gamma)$ is a Mann pair; see Theorem 1.1 of [3].

\(^1\)This is not quite the definition of Mann pair in [3], but is equivalent to it.
Let \( n \geq 2 \) and \( a_1, \ldots, a_n \in k^x \). A solution \( \vec{s} = (s_1, \ldots, s_n) \in (\Omega^x)^n \) of
\[
(*) \quad a_1x + \cdots + a_nx_n = 0
\]
is said to be primitive over \( k \) if \( (s_i)_{i \in I} \) is linearly independent over \( k \) for every nonempty proper subset \( I \) of \( \{1, \ldots, n\} \). So a primitive solution of \((*)\) over \( k \) is in particular a nondegenerate solution of \((*)\).

For \( n \geq 2 \), let \( \Gamma(k, n)_{pr} \) be the set of \( \vec{\gamma} \in \Gamma^n \) such that \( \gamma_n = 1 \) and \( \vec{\gamma} \) is a primitive solution of \((*)\) over \( k \) for some \( a_1, \ldots, a_n \in k^x \); in particular, \( \Gamma(k, n)_{pr} \subseteq \Gamma(k, n) \). In Section 3 we axiomatize the first-order theory of \((\Omega, k, \Gamma)\) when \((k, \Gamma)\) is a Mann pair. More precisely, let \( L \) be the language of rings augmented by two distinct unary relation symbols, and let \( T \) be the \( L\)-theory whose models are the structures \((\Omega, k, \Gamma)\).

**Theorem 1.1.** Suppose \( k \) is infinite, \((k, \Gamma)\) is a Mann pair with \([\Theta : k] > 2\), and \((\Theta', k', \Gamma')\) is a model of \( T \) with \((\Theta, k, \Gamma)\) as a substructure such that \([\Theta' : k'] > 2\), and \( \Gamma'(k', n)_{pr} = \Gamma(k, n)_{pr} \) for every \( n \geq 2 \). Then
\[
(\Theta, k, \Gamma) \equiv_{k, \Gamma} (\Theta, k, \Gamma) \iff k \preceq k' \text{ and } \Gamma \preceq \Gamma'.
\]
This improves on related results from [3] in not involving a choice of finite subset of \( \Gamma^n \) for \( n = 2, 3, \ldots \), nor a choice of basis for the \( k\)-linear spaces attached to the elements of these finite sets.

For Mann pairs \((k, \Gamma)\) the subsets of \( k^m \times \Gamma^m \) that are definable in \((\Omega, k, \Gamma)\) are determined in Proposition 7.2 of [3], but in the present paper we wish to describe more generally the subsets of \( \Omega^m \) definable in \((\Omega, k, \Gamma)\):

**Theorem 1.2.** Suppose \((k, \Gamma)\) is a Mann pair, \( k \) is algebraically closed, \( k \not= \Omega \), and \( \Gamma/\Gamma[p] \) is finite for each prime \( p \). Then a subset of \( \Omega^m \) is definable in \((\Omega, k, \Gamma)\) if and only if it is a boolean combination of subsets of \( \Omega^m \) of the form
\[
\bigcup_{\vec{a} \in k^d} \bigcup_{\vec{\gamma} \in \Gamma^e} X(\vec{a}, \vec{\gamma}), \quad (d, e \in \mathbb{N})
\]
where \( X \subseteq \Omega^{d+e+m} \) is definable in the field \( \Omega \) and \( X(\vec{a}, \vec{\gamma}) \) is the set of all \( \vec{s} \in \Omega^m \) such that \( (\vec{a}, \vec{\gamma}, \vec{s}) \in X \).

In other words, \((\Omega, k, \Gamma)\) eliminates quantifiers down to existential formulas with quantifiers ranging only over \( k \) and \( \Gamma \).

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### 2. Some new Mann pairs

We indicate here some natural Mann pairs that we noticed recently. The first is an “easy” Mann pair in the sense of Section 2.2 of [3], and originates from the following classical result due to E. Borel [1], p. 387:

Let \( f_1, \ldots, f_n \) with \( n \geq 1 \) be entire functions of one complex variable \( z \) that are linearly independent over \( \mathbb{C} \), and suppose each \( f_j \) as well as \( f_1 + \cdots + f_n \)
has only finitely many zeros. Then there is an entire function \( h \) and there are polynomials \( p_j \in \mathbb{C}[z] \) such that \( f_j = p_j e^h \) for \( j = 1, \ldots, n \).

Consider now inside the field of meromorphic functions on the complex plane \( \mathbb{C} \) the subfield \( k = \mathbb{C}(z) \) of rational functions, and the multiplicative group \( \Gamma := \{ e^h : h \text{ is entire and } h(0) = 0 \} \). It is easy to see that Borel’s theorem is equivalent to the proposition that \( (k, \Gamma) \) is an easy Mann pair. The condition \( h(0) = 0 \) is just a normalization to arrange that \( \Gamma \) is torsion-free and \( k^\times \cap \Gamma = \{ 1 \} \).

Next, let \( k \) be any field of characteristic 0. Then the generalized power series field \( K := k((t^Q)) \) in the variable \( t \) comes with a valuation \( v : K^\times \to \mathbb{Q} \), and a derivation \( d/dt \) on \( K \) with constant field \( k \). Let

\[
\mathcal{O} := \{ f \in K : vf \geq 0 \}, \quad \mathfrak{m} := \{ f \in K : vf > 0 \}
\]

be the valuation ring of \( v \) and its maximal ideal. The exponential function

\[
f \mapsto \exp(f) := \sum_{n=0}^{\infty} \frac{f^n}{n!} : \mathfrak{m} \to 1 + \mathfrak{m},
\]

is an isomorphism of the additive group \( \mathfrak{m} \) onto the subgroup \( 1 + \mathfrak{m} \) of \( K^\times \), and \( \exp(f)’ = f’ \exp(f) \) for \( f \in \mathfrak{m} \), with \( g’ := dg/dt \).

**Proposition 2.1.** Let \( L \) be a finite dimensional subspace of the \( k \)-linear space \( \mathfrak{m} \), and put \( \Gamma := \exp L \). Then \( (k, \Gamma) \) is a Mann pair.

**Proof.** We can basically repeat the proof at the end of Section 6 of [3], which is based on Corollary 2.7 in [5]: assume without loss that \( k \) is algebraically closed and extend the differential field \( K \) (with derivation \( d/dt \)) to a differentially closed field \( \Omega \) with constant field \( k \). Consider the logarithmic-derivative map \( \text{ld} : \Omega^\times \to \Omega, \text{ld}(x) := x’/x \). Take \( r \in \mathbb{N} \) and \( f_1, \ldots, f_r \in \mathfrak{m} \) with \( L = k \cdot f_1 + \cdots + k \cdot f_r \) and put \( \gamma_i := \exp f_i \). Then

\[
\text{ld}(\Gamma) = k \cdot f_1’ + \cdots + k \cdot f_r’ = k \cdot \text{ld}(\gamma_1) + \cdots + k \cdot \text{ld}(\gamma_r),
\]

and from here on the proof at the end of Section 6 of [3] goes through word for word.

\( \square \)

This proposition and its proof go through for some other differential fields with a (partial) exponential function, like the field \( \mathbb{R}[[[x]]] \) of transseries.

### 3. Axiomatizing \((\Omega, k, \Gamma)\)

Inspired by Section 6 of [2] we first show that the solutions in \( \Gamma \) of equations

\[
a_1 x_1 + \cdots + a_n x_n = 0, \quad (a_1, \ldots, a_n \in k^\times)
\]

are generated in a certain way by its primitive solutions in \( \Gamma \).
3.1. **Linear considerations.** Below, \( \vec{s} = (s_1, \ldots, s_n) \in (\Omega^x)^n, n \geq 2, \) and \( I \subseteq \{1, \ldots, n\} \). We say that \( I \) is \( \vec{s} \)-minimal over \( k \) if the tuple \((s_i)_{i \in I}\) is linearly dependent over \( k \) (hence \(|I| \geq 2\)), and for each proper subset \( J \) of \( I \) the tuple \((s_j)_{j \in J}\) is linearly independent over \( k \).

Suppose \( I \) is \( \vec{s} \)-minimal over \( k \). Then there is a tuple \((a_i)_{i \in I}\) with all \( a_i \in k \) and some \( a_i \neq 0 \) such that \( \sum_{i \in I} a_i s_i = 0 \); such a tuple \((a_i)\) is unique up to multiplication by a nonzero scalar from \( k \) and has \( a_i \neq 0 \) for all \( i \).

Define \( V(\vec{s}) \) to be the \( k \)-linear subspace of \( k^n \) consisting of all \( \vec{a} = (a_1, \ldots, a_n) \) such that \( a_1 s_1 + \cdots + a_n s_n = 0 \). If \( I \) is \( \vec{s} \)-minimal over \( k \), then \( V((s_i)_{i \in I}) \) is a one-dimensional subspace of \( k^I \). For each \( I \) that is \( \vec{s} \)-minimal over \( k \), fix an element \( a'_I \) of \( k^n \) such that \((a'_i)_{i \in I}\) generates \( V((s_i)_{i \in I}) \) and \( a'_j = 0 \) for \( j \notin I \). The next two lemmas are reformulations of Lemmas 4 and 5 in [2].

**Lemma 3.1.** The \( k \)-linear space \( V(\vec{s}) \) is generated by the \( a'_I \) for which \( I \) is \( \vec{s} \)-minimal over \( k \).

For \( J \subseteq \{1, \ldots, n\} \) we put \( J^c := \{1, \ldots, n\} \setminus J \).

**Lemma 3.2.** Let \( \vec{a} \in V(\vec{s}), J \subseteq \{1, \ldots, n\}, \) and \( \sum_{j \in J} a_j s_j \neq 0 \). Then there is an \( I \) that is \( \vec{s} \)-minimal over \( k \) and meets both \( J \) and \( J^c \).

The direction \((1) \Rightarrow (2)\) of the next lemma is essentially Lemma 6 of [2], but we also need \((2) \Rightarrow (1)\), so we give a complete proof.

**Lemma 3.3.** Assume \( k \) is infinite. Then the following are equivalent:

1. \( \vec{s} \) is a nondegenerate solution of some equation \( a_1 x_1 + \cdots + a_n x_n = 0 \) with \( a_1, \ldots, a_n \in k \);
2. \( \{1, \ldots, n\} \) can be covered by subsets \( I_1, \ldots, I_m \) that are \( \vec{s} \)-minimal over \( k \), such that for every nonempty proper subset \( J \) of \( \{1, \ldots, n\} \) some \( I_p \) with \( p \in \{1, \ldots, m\} \) meets both \( J \) and \( J^c \).

**Proof.** To show that \((1) \) implies \((2)\), let \( a_1, \ldots, a_n \in k^x \) be such that \( \vec{s} \) is a nondegenerate solution of \( a_1 x_1 + \cdots + a_n x_n = 0 \). Take an \( I_1 \subseteq \{1, \ldots, n\} \) that is \( \vec{s} \)-minimal over \( k \). Note that if \( I_1 = \{1, \ldots, n\} \), then \((2)\) holds with \( m = 1 \). Suppose \( I_1, \ldots, I_m \subseteq \{1, \ldots, n\} \) with \( m \geq 1 \) are \( \vec{s} \)-minimal over \( k \), we have a strictly increasing chain

\[
I_1 \subseteq I_1 \cup I_2 \subseteq \cdots \subseteq I_1 \cup \cdots \cup I_m,
\]

and for every nonempty proper subset \( J \) of \( I := I_1 \cup \cdots \cup I_m \) some \( I_p \) with \( p \in \{1, \ldots, m\} \) meets both \( J \) and \( J^c \). If \( I = \{1, \ldots, n\} \), then \((2)\) holds. Assume \( I \neq \{1, \ldots, n\} \). Then \( \sum_{i \in I} a_i s_i \neq 0 \), so by Lemma 3.2 we have an \( I_{m+1} \subseteq \{1, \ldots, n\} \) that is \( \vec{s} \)-minimal over \( k \) and meets both \( I \) and \( I^c \).

It follows easily that then for every nonempty proper subset \( J \) of \( I \cup I_{m+1} \) some \( I_p \) with \( p \in \{1, \ldots, m+1\} \) meets both \( J \) and \( J^c \). So in a finite number of steps we obtain a covering as in \((2)\).
To prove the converse, assume (2). Take $I_1, \ldots, I_m$ as in (2), and for $p = 1, \ldots, m$, take $a_{pj} \in k$, $j = 1, \ldots, n$, such that
\[
\sum_{j=1}^{n} a_{pj}s_j = 0, \quad a_{pj} = 0 \text{ for } j \notin I_p, \quad a_{pj} \neq 0 \text{ for } j \in I_p.
\]
With $x_1, \ldots, x_m \in k$ we have
\[
0 = x_1 \sum_{j=1}^{n} a_{1j}s_j + \cdots + x_m \sum_{j} a_{mj}s_j = (\sum_{p=1}^{m} a_{p1}x_1)s_1 + \cdots + (\sum_{p=1}^{m} a_{pn}x_n)s_n
\]
Thus it suffices to find $x_1, \ldots, x_m \in k$ such that for each nonempty proper subset $J$ of $\{1, \ldots, n\}$ we have $\sum_{j \in J}(\sum_{p=1}^{m} a_{pj}x_i)s_j \neq 0$, that is,
\[
(\sum_{j \in J} a_{1j}s_j)x_1 + \cdots + (\sum_{j \in J} a_{mj}s_j)x_m \neq 0.
\]
For each nonempty proper subset $J$ of $\{1, \ldots, n\}$ we take $p \in \{1, \ldots, m\}$ such that $I_p$ meets both $J$ and $J^c$, and so $\sum_{j \in J} a_{pj}s_j \neq 0$. Since $k$ is infinite, this yields $x_1, \ldots, x_m \in k$ as desired. $\square$

Lemma 3.3 says basically how $\Gamma(k, n)$ is determined by the sets $\Gamma(k, m)^{pr}$ with $m = 2, \ldots, n$. Here are some consequences:

**Corollary 3.4.** $(k, \Gamma)$ is a Mann pair iff $\Gamma(k, n)^{pr}$ is finite for all $n \geq 2$.

*Proof.* Let $n \geq 2$ be given and assume $\Gamma(k, m)^{pr}$ is finite for $m = 2, \ldots, n$. We shall derive that $\Gamma(k, n)$ is finite. Let $\gamma \in \Gamma(k, n)$. The proof of the direction (1) $\Rightarrow$ (2) of Lemma 3.3 does not use that $k$ is infinite, so we have a covering of $\{1, \ldots, n\}$ by subsets $I_1, \ldots, I_m$ that are $\gamma$-minimal over $k$, such that for every nonempty proper subset $J$ of $\{1, \ldots, n\}$ some $I_p$ meets both $J$ and $J^c$. By renumbering the $I$’s we arrange that $n \in I_1$, and since $\gamma_n = 1$ this leaves only finitely many possibilities for $(\gamma_i)_{i \in I_1}$. If $I_1 = \{1, \ldots, n\}$ we are done. Otherwise, we can assume that $I_2$ meets both $I_1$ and $I_1^c$. Taking $i_1 \in I_1 \cap I_2$ we have only finitely many possibilities for $\gamma_{i_1}$, and so there are only finitely many possibilities for $(\gamma_i)_{i \in I_2}$ and thus for $(\gamma_i)_{i \in I_1 \cup I_2}$. If $I_1 \cup I_2 = \{1, \ldots, n\}$ we are done, and otherwise we continue as above. $\square$

**Corollary 3.5.** Suppose $k$ is infinite and $K \supseteq k$ is a subfield of $\Omega$ such that $\Gamma(k, n)^{pr} = \Gamma(K, n)^{pr}$ for all $n \geq 2$. Then $\Gamma(k, n) = \Gamma(K, n)$ for all $n \geq 2$.

*Proof.* Let $n \geq 2$ and $\gamma \in \Gamma(k, n)$. Then the direction (1) $\Rightarrow$ (2) of Lemma 3.3 yields a covering of $\{1, \ldots, n\}$ by subsets $I_1, \ldots, I_m$ that are $\gamma$-minimal over $k$, such that for every nonempty proper subset $J$ of $\{1, \ldots, n\}$ some $I_p$ with $p \in \{1, \ldots, m\}$ meets both $J$ and $J^c$. Then $I_1, \ldots, I_m$ are also $\gamma$-minimal over $K$, so by the direction (2) $\Rightarrow$ (1) of Lemma 3.3 we have $\gamma \in \Gamma(K, n)$. $\square$
This gives an improvement of (4) in Section 5 of [3] for infinite \( k \):

**Corollary 3.6.** Suppose \( k \) is infinite and \( K \supseteq k \) is subfield of \( \Omega \) that is linearly disjoint from \( k(\Gamma) \) over \( k \). Then \( \Gamma(k, n) = \Gamma(K, n) \) for all \( n \geq 2 \).

*Proof.* The linear disjointness assumption yields \( \Gamma(k, n) = \Gamma(K, n) \) for all \( n \geq 2 \). Now use Corollary 3.5. \( \square \)

### 3.2. Elementary classification.

Consider a model \((\Omega_0, k_0, \Gamma_0)\) of \( T \) such that \( k_0 \) is infinite and \((k_0, \Gamma_0)\) is a Mann pair. We now have Theorem 1.1 in the following stronger form.

**Theorem 3.7.** Let \((\Omega_1, k_1, \Gamma_1)\) and \((\Omega_2, k_2, \Gamma_2)\) be models of \( T \) such that

1. \( |\Omega_1 : k_1| > 2 \) and \( |\Omega_2 : k_2| > 2 \);
2. \( (\Omega_0, k_0, \Gamma_0) \subseteq (\Omega_1, k_1, \Gamma_1) \) and \((\Omega_0, k_0, \Gamma_0) \subseteq (\Omega_2, k_2, \Gamma_2)\);
3. \( \Gamma_1(k_1, n)pr = \Gamma_2(k_2, n)pr = \Gamma_0(k_0, n)pr \) for all \( n \geq 2 \).

Then: \((\Omega_1, k_1, \Gamma_1) \equiv_{k_0, \Gamma_0} (\Omega_2, k_2, \Gamma_2) \iff k_1 \equiv_{k_0} k_2 \) and \( \Gamma_1 \equiv_{\Gamma_0} \Gamma_2 \).

*Proof.* It follows easily from Lemma 3.1 and Corollary 3.5 that \((\Omega_i, k_i, \Gamma_i)\) satisfies the Mann axioms of \((\Omega_0, k_0, \Gamma_0)\) for \( i = 1, 2 \), in the sense of [3]. It remains to use Theorem 8.4 in [3]. \( \square \)

For algebraically closed \( k_i \) this gives:

**Corollary 3.8.** Let \((\Omega_1, k_1, \Gamma_1)\) and \((\Omega_2, k_2, \Gamma_2)\) be models of \( T \) such that

1. \( k_1 \) and \( k_2 \) are algebraically closed, \( k_1 \neq \Omega_1 \), \( k_2 \neq \Omega_2 \);
2. \( (\Omega_0, k_0, \Gamma_0) \subseteq (\Omega_1, k_1, \Gamma_1) \) and \((\Omega_0, k_0, \Gamma_0) \subseteq (\Omega_2, k_2, \Gamma_2)\);
3. \( \Gamma_1(k_1, n)pr = \Gamma_2(k_2, n)pr = \Gamma_0(k_0, n)pr \) for every \( n \geq 2 \).

Then: \((\Omega_1, k_1, \Gamma_1) \equiv_{k_0, \Gamma_0} (\Omega_2, k_2, \Gamma_2) \iff \Gamma_1 \equiv_{\Gamma_0} \Gamma_2 \).

### 4. Definable sets in \((\Omega, k, \Gamma)\)

**A back-and-forth system.** To define a back-and-forth system adequate for proving Theorem 1.2 is not so obvious, and we managed to do it only after much trial-and-error; see Lemmas 4.1, 4.2, 4.3. Throughout this subsection we fix a model \((\Omega_0, k_0, \Gamma_0)\) of \( T \) such that \( k_0 \) is infinite and \((k_0, \Gamma_0)\) is a Mann pair.

**Lemma 4.1.** Let \((\Omega, k, \Gamma)\) be a model of \( T \) such that \((\Omega_0, k_0, \Gamma_0) \subseteq (\Omega, k, \Gamma)\) and \( \Gamma(k, n)pr = \Gamma_0(k_0, n)pr \) for every \( n \geq 2 \). Then

1. \( k \) and \( k_0(\Gamma) \) are linearly disjoint over \( k_0 \);
2. if \( k|k_0 \) is regular and \( \Gamma|\Gamma_0 \) is pure, then \( k(\Gamma)|k_0(\Gamma_0) \) is regular.

*Proof.* Let \( \gamma_1, \ldots, \gamma_n \in \Gamma \) be linearly dependent over \( k \); to get (1) it is enough to show that then they are linearly dependent over \( k_0 \). We can reduce to the case that \( n \geq 2 \) and \( (\gamma_i)_{i \in I} \) is linearly independent over \( k \) for all nonempty proper subsets \( I \) of \( \{1, \ldots, n\} \). Then by Corollary 3.5 we have

\[
(\gamma_1, \ldots, \gamma_n) = \gamma \gamma_0, \quad \gamma \in \Gamma, \quad \gamma_0 \in \Gamma_0(k_0, n)pr,
\]
so \( \gamma_1, \ldots, \gamma_n \) are indeed linearly dependent over \( k_0 \).

Suppose now that \( k|k_0 \) is regular and \( \Gamma|\Gamma_0 \) is pure. Then by (1) and Theorem 4.13 from [6] the extension \( k(\Gamma)|k_0(\Gamma) \) is regular, and by Lemma 5.13 of [4] the extension \( k_0(\Gamma)|k_0(\Gamma_0) \) is regular. Hence \( k(\Gamma)|k_0(\Gamma_0) \) is regular, by Proposition 4.11(b) of Chapter VIII of [6].

Fix a cardinal \( \kappa > |\Omega_0| \), and let \((\Omega, k, \Gamma)\) be a \( \kappa \)-saturated model of \( T \) such that \((\Omega_0, k_0, \Gamma_0) \subseteq (\Omega, k, \Gamma) \) and \( \Gamma(k, n)^{pr} = \Gamma_0(k_0, n)^{pr} \) for every \( n \geq 2 \). Define \( \text{Sub}(\Omega, k, \Gamma) \) to be the collection of all models \((\Omega', k', \Gamma')\) of \( T \) such that:

(i) \( \Omega_0 \subseteq \Omega' \subseteq \Omega \) (as fields) and \( |\Omega'| < \kappa \);
(ii) \( k_0 \subseteq k' \subseteq k \) (as fields) and the extension \( k|k' \) is regular;
(iii) \( \Gamma_0 \subseteq \Gamma' \subseteq \Gamma \) (as groups), and the extension \( \Gamma|\Gamma' \) is pure;
(iv) \( k(\Gamma) \) and \( \Omega' \) are free over \( k'(\Gamma') \).

So \( (\Omega', k', \Gamma') \in \text{Sub}(\Omega, k, \Gamma) \) yields a diagram of field inclusions:

\[
\begin{array}{ccc}
\Omega & \quad k(\Gamma) & \quad \Omega' \\
& k & \quad k'(\Gamma') \\
\end{array}
\]

**Lemma 4.2.** Let \( (\Omega', k', \Gamma') \in \text{Sub}(\Omega, k, \Gamma) \). Then

1. \( k'(\Gamma') \) and \( \Omega' \) are linearly disjoint over \( k'(\Gamma') \),
2. \( k \) and \( \Omega' \) are linearly disjoint over \( k' \),
3. \( k(\Gamma') \cap \Gamma = \Gamma' \),
4. \( (\Omega', k', \Gamma') \) is a substructure of \((\Omega, k, \Gamma)\).

**Proof.** From the linear disjointness of \( k \) and \( k_0(\Gamma) \) over \( k_0 \)—item (1) of Lemma 4.1—we obtain the linear disjointness of \( k \) and \( k'(\Gamma') \) over \( k' \). Hence \( k(\Gamma')|k'(\Gamma') \) is regular. We then argue as in the proof of (2) in Lemma 4.1, with \( k' \) instead of \( k_0 \), that \( k'(\Gamma)|k'(\Gamma') \) is regular, and so \( k(\Gamma)|k'(\Gamma') \) is regular. Then (1) follows from condition (iv) above and Theorem 4.12 on page 367 of [6]. This also gives (2).

For (3), let \( \gamma \in k(\Gamma') \cap \Gamma', \) so \( \gamma = \sum a_1 \alpha_1 + \cdots + a_m \alpha_m + b_1 \beta_1 + \cdots + b_n \beta_n \) with \( a_1, \ldots, a_m, b_1, \ldots, b_n \) from \( k \), and \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \) from \( \Gamma' \), and \( b_1 \beta_1 + \cdots + b_n \beta_n \neq 0 \). Taking such a representation of \( \gamma \) with minimal \( m + n \), we have \( m, n \geq 1 \), and using Corollary 3.5, this gives

\[
(\alpha_1, \ldots, \alpha_m, \beta_1 \gamma, \ldots, \beta_n \gamma) \in \Gamma \bar{\gamma}_0, \quad \bar{\gamma}_0 \in \Gamma_0(k_0, m + n).
\]
In particular, \( \frac{\partial \gamma}{\partial n} \in \Gamma_0 \subseteq \Gamma' \), so \( \gamma \in \Gamma' \).

As to (4), by (2) we have \( \Omega' \cap k = k' \), so if \( \gamma \in \Omega' \cap \Gamma \), then \( \gamma \in k'(\Gamma') \) by (1), and hence \( \gamma \in \Gamma' \) by (3).

Next, for \( i = 1, 2 \), let \( (\Omega_i, k_i, \Gamma_i) \) be a \( \kappa \)-saturated model of \( T \) such that \((\Omega_0, k_0, \Gamma_0) \subseteq (\Omega_i, k_i, \Gamma_i), [\Omega_i : k_i] > 2 \), and \( \Gamma_i(k_i, n)_{\text{pr}} = \Gamma_0(k_0, n)_{\text{pr}} \) for every \( n \geq 2 \), and put \( \text{Sub}_i := \text{Sub}(\Omega_i, k_i, \Gamma_i) \).

Let \( \mathcal{I} \) be the set of isomorphisms

\[ \iota : (\Omega'_1, k'_1, \Gamma'_1) \rightarrow (\Omega'_2, k'_2, \Gamma'_2), \quad (\Omega'_i, k'_i, \Gamma'_i) \in \text{Sub}_i \text{ for } i = 1, 2, \]

that are the identity on \( k_0 \) and on \( \Gamma_0 \), such that \( \iota|k'_i \) is a partial elementary map from \( k_1 \) to \( k_2 \), and \( \iota|\Gamma'_i \) is a partial elementary map from \( \Gamma_1 \) to \( \Gamma_2 \); we do not require that \( \iota \) is the identity on \( \Omega_0 \).

**Lemma 4.3.** \( \mathcal{I} \) is a (possibly empty) back-and-forth system.

**Proof.** Let \( \iota : (\Omega'_1, k'_1, \Gamma'_1) \rightarrow (\Omega'_2, k'_2, \Gamma'_2) \) be in \( \mathcal{I} \), and \( r \in \Omega_1 \setminus \Omega_1' \); by symmetry it is enough to show that then \( \iota \) extends to an isomorphism in \( \mathcal{I} \) that has \( r \) in its domain.

First, consider the case that \( r \in k_1 \). Then we take a field \( k''_1 \leq k_1 \) such that \( k'_1(r) \subseteq k''_1 \) and \( [k''_1 : k'_1] < \kappa \). Using saturation we extend \( \iota|k'_1 \) to a field isomorphism \( f : k''_1 \rightarrow k''_2 \leq k_2 \) that is a partial elementary map between \( k_1 \) and \( k_2 \). It is clear that \( k''_1(k''_1(r)) \) is regular, and that \( k''_1(k''_1(r)) \) and \( k''_2(k''_1(r)) \) are free over \( k''_1(\Gamma''_1) \), where \( \Omega''_i := (\Omega'_i(k''_i))^{\text{ac}} \) and \( i \in \{1, 2\} \). So \( (\Omega''_1, k''_1, \Gamma''_1) \in \text{Sub}_i \) for \( i = 1, 2 \). Then by (2) of Lemma 4.2 we have a common extension of \( \iota \) and \( f \) to an isomorphism

\[ (\Omega'_1, k'_1, \Gamma'_1) \rightarrow (\Omega''_1, k''_1, \Gamma''_1) \]

in \( \mathcal{I} \); it has \( r \) in its domain.

Next, assume that \( r \in \Gamma_1 \). Then we take a group \( \Gamma''_1 \leq \Gamma_1 \) such that \( r^\mathcal{I}\Gamma''_1 \subseteq \Gamma''_1 \) and \( [\Gamma''_1 : \Gamma_1] < \kappa \). Using saturation we extend \( \iota|\Gamma_1' \) to a group isomorphism \( g : \Gamma''_1 \rightarrow \Gamma''_2 \leq \Gamma_2 \) that is a partial elementary map between \( \Gamma_1 \) and \( \Gamma_2 \). It is clear that \( \Gamma_i(\Gamma_i') \) is pure, and that \( k_i(\Gamma_i) \) and \( \Omega''_i \) are free over \( k'_i(\Gamma''_i) \), where \( \Omega''_i := (\Omega'_i(\Gamma''_i))^{\text{ac}} \) and \( i \in \{1, 2\} \). So \( (\Omega''_i, k''_i, \Gamma''_i) \in \text{Sub}_i \) for \( i = 1, 2 \). Also \( \Gamma_i(k_i, n) = \Gamma_0(k_0, n) \), for \( n > 1 \) and \( i = 1, 2 \), and thus we have a field isomorphism \( h : k'_i(\Gamma''_i) \rightarrow k''_i(\Gamma''_i) \) that extends \( \iota|k'_i \) and \( g \).

Then by (1) of Lemma 4.2 this gives a common extension of \( \iota \) and \( h \) to an isomorphism

\[ (\Omega'_1, k'_1, \Gamma'_1) \rightarrow (\Omega''_2, k''_2, \Gamma''_2) \]

in \( \mathcal{I} \); it has \( r \) in its domain.

If \( r \in \Omega'_i(k_1 \cup \Gamma_1)^{\text{ac}} \), then we can take a finite number of steps of the two types above to extend \( \iota \) to an element of \( \mathcal{I} \) with \( r \) in its domain.

Finally, suppose that \( r \notin \Omega'_i(k_1 \cup \Gamma_1)^{\text{ac}} \). By saturation and the smallness assumption \([\Omega_2 : k_2] > 2\) we can take \( s \in \Omega_2 \) with \( s \notin \Omega'_2(k_2 \cup \Gamma_2)^{\text{ac}} \). With

\[ \Omega'_i := \Omega'_i(\gamma)^{\text{ac}}, \quad \Omega''_2 := \Omega'_2(s)^{\text{ac}}, \]

we can take a sequence of isomorphisms

\[ (\Omega'_i, k'_i, \Gamma'_i) \rightarrow (\Omega''_i, k''_i, \Gamma''_i) \]

in \( \mathcal{I} \); it has \( r \) in its domain.
it is clear that \((\Omega'', k'_i, \Gamma''_i) \in \text{Sub}_t\) for \(i = 1, 2\). We can extend \(t\) to a field isomorphism \(\Omega''_1 \rightarrow \Omega''_2\) that sends \(r\) to \(s\), and this gives an isomorphism
\[
(\Omega''_{1, k'_1, \Gamma''_1}) \rightarrow (\Omega''_{2, k'_2, \Gamma''_2})
\]
in \(I\) with \(r\) in its domain.

\[\square\]

**Corollary 4.4.** Suppose that \((\Omega_0, k_0, \Gamma_0) \subseteq (\Omega, k, \Gamma) \models T\) and \([\Omega_0 : k_0] > 2\). Then \((\Omega_0, k_0, \Gamma_0) \preceq (\Omega, k, \Gamma)\) if and only if (1) – (4) below are satisfied:

1. \([\Omega : k] > 2\);
2. \(\Gamma(k, n)^{pr} = \Gamma_0(k_0, n)^{pr}\) for every \(n \geq 2\);
3. \(k_0 \preceq k\) and \(\Gamma_0 \preceq \Gamma\);
4. \(k(\Gamma)\) and \(\Omega_0\) are free over \(k_0(\Gamma_0)\).

**Proof.** It is easy to check that if \((\Omega_0, k_0, \Gamma_0) \preceq (\Omega, k, \Gamma)\), then (1)–(4) hold. For the converse, assume (1)–(4). By passing to an elementary extension of \((\Omega, k, \Gamma)\) we arrange that \((\Omega, k, \Gamma)\) is \(\kappa\)-saturated. Put
\[
(\Omega_1, k_1, \Gamma_1) := (\Omega, k, \Gamma)
\]
and let \((\Omega_2, k_2, \Gamma_2)\) be a \(\kappa\)-saturated elementary extension of \((\Omega_0, k_0, \Gamma_0)\). Then \((\Omega_0, k_0, \Gamma_0) \in \text{Sub}(\Omega_1, k_1, \Gamma_1)\) for \(i = 1, 2\). Let \(I\) be the back-and-forth system considered in the previous lemma. Then the identity map on \((\Omega_0, k_0, \Gamma_0)\) belongs to \(I\), and so \((\Omega_1, k_1, \Gamma_1)\) and \((\Omega_2, k_2, \Gamma_2)\) are elementarily equivalent over \(\Omega_0\). Thus \((\Omega_0, k_0, \Gamma_0) \preceq (\Omega, k, \Gamma)\). \[\square\]

**Definable Sets.** We now specify the two unary relation symbols of \(L\) to be \(U\) and \(V\), to be interpreted in a model of \(T\) as the underlying set of the distinguished subfield and of the distinguished multiplicative group, respectively. Let a model \((\Omega, k, \Gamma)\) of \(T\) be given and let \(x = (x_1, \ldots, x_m)\) be a tuple of distinct variables. Call a subset of \(\Omega^m\) special if it is defined in \((\Omega, k, \Gamma)\) by a special formula in \(x = (x_1, \ldots, x_m)\), that is, a formula
\[
\exists y \exists z (U(y) \land V(z) \land \phi(x, y, z)),
\]
where \(x_1, \ldots, x_m, y_1, \ldots, y_s, z_1, \ldots, z_t\) are distinct variables, \(y = (y_1, \ldots, y_s)\), \(z = (z_1, \ldots, z_t)\) and \(\phi(x, y, z)\) is a quantifier-free formula in the language of rings augmented by names for the elements of \(\Omega\), and where \(U(y)\) and \(V(z)\) abbreviate \(U(y_1) \land \cdots \land U(y_s)\) and \(V(z_1) \land \cdots \land V(z_t)\), respectively.

Now we are ready to prove Theorem 1.2, which we first reformulate using the above terminology.

**Theorem 4.5.** Let \((\Omega, k, \Gamma)\) be a model of \(T\) such that \(k\) is an algebraically closed field, \(k \neq \Omega\), \((k, \Gamma)\) is a Mann pair, and \(\Gamma/\Gamma^{[p]}\) is finite for each \(p\). Then the subsets of \(\Omega^m\) definable in \((\Omega, k, \Gamma)\) are exactly the boolean combinations in \(\Omega^m\) of special subsets of \(\Omega^m\).

**Proof.** We take \(\kappa := \aleph_1\) and may assume that \((\Omega, k, \Gamma)\) is \(\kappa\)-saturated. Let \((\Omega'_0, k_0, \Gamma_0)\) be a countable elementary substructure of \((\Omega, k, \Gamma)\), and let \(\Omega_0\) be the algebraic closure of \(k_0(\Gamma_0)\) in \(\Omega'_0\). Then \((\Omega_0, k_0, \Gamma_0) \subseteq (\Omega, k, \Gamma)\),...
\((k_0, \Gamma_0)\) is a Mann pair, and \(\Gamma(k, n)^{pr} = \Gamma_0(k_0, n)^{pr}\) for every \(n \geq 1\). Let \(\vec{r} = (r_1, \ldots, r_m)\) and \(\vec{s} = (s_1, \ldots, s_m)\) be two tuples from \(\Omega^m\) that satisfy the same special formulas in \(x\) using only names for elements of \(A := k_0 \cup \Gamma_0\); it suffices to show that then they realize the same type in \((\Omega, k, \Gamma)\) over \(A\).

For \(i = 1, 2\), put \((\Omega_i, k_i, \Gamma_i) := (\Omega, k, \Gamma)\). Hence all the structures in \(\text{Sub}(\Omega_i, k_i, \Gamma_i)\) are countable. Set \(\mathcal{I}\) to be the back-and-forth system of Lemma 4.3; it is enough to construct an isomorphism in this system that takes \(\vec{r}\) to \(\vec{s}\).

Let \(d\) be the transcendence degree of \(k(\Gamma)(\vec{r})\) over \(k(\Gamma)\). We can assume that \(r_1, \ldots, r_d\) are algebraically independent over \(k(\Gamma)\). As in the proof of Theorem 3.8 in [4] it follows that \(s_1, \ldots, s_d\) are algebraically independent over \(k(\Gamma)\), and \(s_{d+1}, \ldots, s_m\) are algebraic over \(k(\Gamma)(s_1, \ldots, s_d)\).

Take some countable \((\Omega', k', \Gamma') \preceq (\Omega, k, \Gamma)\) such that \(\Omega' \supseteq \Omega_0\) and \(k'(\Gamma')(\vec{r})\) has transcendence degree \(d\) over \(k'(\Gamma')\). In particular, \((\Omega', k', \Gamma') \in \text{Sub}(\Omega, k, \Gamma)\). Let \(a = (a_0, a_1, a_2, \ldots)\) be an enumeration of \(k'\), let \(g = (g_0, g_1, g_2, \ldots)\) be an enumeration of \(\Gamma'\), and let \(y_0, y_1, y_2, \ldots\) be distinct variables, also distinct from \(x_1, \ldots, x_m\), and put

\[
y = (y_0, y_1, y_2, \ldots), \quad z = (z_0, z_1, z_2, \ldots).
\]

Suppose \(\psi_1(y), \ldots, \psi_k(y)\) are quantifier-free formulas in the language of rings augmented by names for the elements of \(k_0\), and \(\theta_1(z), \ldots, \theta_k(z)\) are quantifier-free formulas in the language of groups augmented by names for the elements of \(\Gamma_0\), and \(\phi_1(x, y, z), \ldots, \phi_k(x, y, z)\) are quantifier-free formulas in the language of rings augmented by names for the elements of \(A\), such that \(k \models \psi_j(a), \Gamma \models \theta_j(g)\) and \((\Omega, k, \Gamma) \models \phi_j(\vec{r}, a, g)\) for \(j = 1, \ldots, k\). Then

\[
(\Omega, k, \Gamma) \models \exists y \exists z (U(y) \land V(z) \land \psi(y) \land \theta(z) \land \phi(\vec{r}, y, z)),
\]

where

\[
\psi(y) := \bigwedge_j \psi_j(y), \quad \theta(z) := \bigwedge_j \theta_j(z), \quad \varphi(x, y, z) := \bigwedge_j \varphi_j(x, y, z).
\]

The assumption on \(\vec{r}\) and \(\vec{s}\) then gives

\[
(\Omega, k, \Gamma) \models \exists y \exists z (U(y) \land V(z) \land \psi(y) \land \theta(z) \land \phi(\vec{s}, y, z)).
\]

Hence we have a partial \(y, z\)-type over \(A\vec{s}\) in \((\Omega, k, \Gamma)\) consisting of the formulas \(U(y_i)\) and \(V(z_i)\) for \(i = 0, 1, 2, \ldots\), the quantifier-free formulas \(\psi(y)\) in the language of rings augmented by names for the elements of \(k_0\) such that \(k \models \psi(a)\), the quantifier-free formulas \(\theta(z)\) in the language of groups augmented by names for the elements of \(\Gamma_0\) such that \(\Gamma \models \theta(g)\), and the formulas \(\phi(\vec{s}, y, z)\) such that \(\phi(x, y, z)\) is a quantifier-free formula in the language of rings augmented by names for the elements of \(A\) and \((\Omega, k, \Gamma) \models \phi(\vec{r}, a, g)\).

Let \(b, h\) with \(b = (b_0, b_1, b_2, \ldots) \in k^\mathbb{N}\) and \(h = (h_0, h_1, h_2, \ldots) \in \Gamma^\mathbb{N}\) realize this \(y, z\)-type in \((\Omega, k, \Gamma)\). Then \(\{b_0, b_1, b_2, \ldots\}\) is the underlying set of a field \(k'' \preceq k\) and we have a field isomorphism

\[
i^f : k' \to k'', \quad i^f(a_n) = b_n \text{ for all } n.
\]
Likewise, \( \{h_0, h_1, h_2, \ldots \} \) is the underlying set of a group \( \Gamma'' \preceq \Gamma \) and we have a group isomorphism
\[
\iota^g : \Gamma' \to \Gamma'', \quad \iota^g(g_n) = h_n \text{ for all } n.
\]
Note that \( \iota^f \) is a partial elementary map from \( k \) to itself and is the identity on \( k_0 \). Likewise, \( \iota^g \) is a partial elementary map from \( \Gamma \) to itself and is the identity on \( \Gamma_0 \). Moreover, \( \iota^f \) and \( \iota^g \) have a common extension to a field isomorphism
\[
\iota : k'(\Gamma')((\vec{r})) \cong k''(\Gamma'')((\vec{s})^\ac)
\]
sending \( \vec{r} \) to \( \vec{s} \). Put \( \Omega'_1 := k'(\Gamma')((\vec{r})^\ac) \) and \( \Omega'_2 := k''(\Gamma'')((\vec{s})^\ac) \). Then
\[
(\Omega'_1, k', \Gamma'), (\Omega'_2, k'', \Gamma'') \in \text{Sub}(\Omega, k, \Gamma),
\]
and we have an isomorphism
\[
(\Omega'_1, k', \Gamma') \cong (\Omega'_2, k'', \Gamma'')
\]
that extends \( \iota \). It carries \( \vec{r} \) to \( \vec{s} \) and belongs to \( \mathcal{I} \).

Note that if \( \Gamma \) is divisible or of finite rank, then the condition in the theorem that \( \Gamma/\Gamma[p] \) is finite for each \( p \) is satisfied.

References