A TALK ON EXPONENTIAL POLYNOMIALS

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INTRODUCTION

We are going to prove the following.

**Theorem.**(Assuming Schanuel’s conjecture) Let \( p(X, Y) \in \mathbb{C}[X,Y] \) be irreducible such that both \( X \) and \( Y \) appear in \( p \). Then there is a generic point on the curve defined by \( p \) of the form \((z, \exp(z))\).

In order to prove this it is enough to show the following. (It is not a trivial reduction; the reader should look at the paper for the details.)

**Proposition.**(Assuming Schanuel’s conjecture) Let \( p(X, Y) \in \mathbb{C}[X,Y] \) be irreducible such that both \( X \) and \( Y \) appear in \( p \) and let \( K \) be an algebraically closed subfield of \( \mathbb{C} \) of finite transcendence degree. Then there are finitely many \( z \in K \) such that \( p(z, \exp(z)) = 0 \).

1. Linear equations in fields of finite transcendence degree

In this section let \( K \) be an algebraically closed subfield of \( \mathbb{C} \) of finite transcendence degree containing \( \pi \) and let \( \Gamma := \exp(K) \). Here we consider solutions in \( \Gamma \) of \(\lambda_1 x_1 + \cdots + \lambda_k x_k = 1\), where \(\lambda_1, \ldots, \lambda_k \in K\). We say that a solution \( \vec{\gamma} = (\gamma_1, \ldots, \gamma_k) \) in \( \Gamma \) of \((\ast)\) is non-degenerate if \(\sum_{I \subseteq \{1, \ldots, k\}} \lambda_I \gamma_I \neq 0 \) for every nonempty proper subset \( I \) of \( \{1, \ldots, k\} \).

We begin with some notations that will be useful.

**Definition 1.1.** Let \( G \) be an abelian group, written multiplicatively and for \( n > 0 \) put \( G^{[n]} = \{g^n : g \in G\} \). We say that a subgroup \( H \) is pure in \( G \) if \( H \cap G^{[n]} = H^{[n]} \) for all \( n > 0 \). We say that \( H \) is radical in \( G \) if it is pure in \( G \) and it contains all the torsion elements of \( G \).

Given \( A \subseteq G \), we set \( \langle A \rangle_G \) to be the smallest radical subgroup of \( G \) containing \( A \). That is,

\[ \langle A \rangle_G = \{ g \in G : g^n \in [A]_G \text{ for some } n \in \mathbb{N} \} \]

where \( [A]_G \) is the subgroup generated by \( A \). When \( G \) is clear from the context, we will drop the subscripts and just write \( \langle A \rangle \) and \( [A] \). For instance throughout this section the ambient group is \( \mathbb{C}^\times \) unless explicitly stated otherwise.

Since \( K \) is a field of characteristic 0, it is easy to see that \( \Gamma \) is divisible; in particular it is pure in \( \mathbb{C}^\times \). Moreover \( \Gamma \) is a radical subgroup of \( \mathbb{C}^\times \) since \( \sqrt{-1}\pi \) is in \( K \). Therefore, so is \( \Gamma \cap K^\times \).
Given $a_1, \ldots, a_n$ in $\mathbb{C}$, by $\vec{a}$ we denote the tuple $(a_1, \ldots, a_n)$ and $\exp(\vec{a})$ denotes $(\exp(a_1), \ldots, \exp(a_n))$.

Note the following straightforward consequence of Schanuel’s Conjecture concerning the rank of $\Gamma \cap K^\times$.

**Lemma 1.2.** (Assuming Schanuel’s conjecture)

The rank of $\Gamma \cap K^\times$ is at most $d$.

On the basis of this lemma, take $\beta_1, \ldots, \beta_t \in K$ where $t \leq d$ such that $\pi\sqrt{-1}, \beta_1, \ldots, \beta_t$ are $\mathbb{Q}$-linearly independent and

$$\Gamma \cap K^\times = \langle \exp(\beta_1), \ldots, \exp(\beta_t) \rangle.$$ 

Recall Lemma 8.2 from [2].

**Lemma 1.3.** Let $F$ be a field with a subfield $E$ and subgroups $G, H$ of $F^\times$. Suppose also that $H$ is a radical subgroup of $G$. Then the following two conditions are equivalent:

1. For every $\lambda_1, \ldots, \lambda_k \in E$, the equation $[x^a]$ has the same non-degenerate solutions in $H$ as in $G$.
2. Whenever $g_1, \ldots, g_n$ in $G$ are multiplicatively independent over $H$, they are algebraically independent over the field $E(H)$.

This allows us to prove the following.

**Proposition 1.4.** (Assuming Schanuel’s conjecture)

There exists a radical subgroup $\Gamma^*$ of $\Gamma$ of finite rank containing $\Gamma \cap K^\times$ such that for every $\lambda_1, \ldots, \lambda_k \in K$, the equation $[x^a]$ has the same non-degenerate solutions in $\Gamma^*$ as in $\Gamma$.

**Remark.** It follows from the proof above that if the rank of $\Gamma \cap K^\times$ is already $d$, then we can take $\Gamma^*$ to be $\Gamma \cap K^\times$.

Let $\Gamma^* = \langle \exp(a_1), \ldots, \exp(a_s) \rangle$ with $a_1, \ldots, a_s \in K$ linearly independent over $\sqrt{-1}\pi$.

From now on, $\mathbb{U}$ denotes the multiplicative group of all roots of unity. Recall the following results.

**Lemma 1.5.** (Lemma 6.1 in [3])

Let $E \subseteq F$ be fields such that $E \cap \mathbb{U} = F \cap \mathbb{U}$ and $G$ be a pure subgroup of $E^\times$. Then for $\lambda_1, \ldots, \lambda_n \in E^\times$, the equation $[x^a]$ has the same non-degenerate solutions in $G$ as in $(G)_{F^\times}$.

**Lemma 1.6.** (Proposition 2.2 (ii) in [10])

Let $L$ be a finitely generated extension of $\mathbb{Q}(\mathbb{U})$. Then the quotient group $L^\times / \mathbb{U}$ is free.

We can now reduce our situation from $K$ to any subfield $L$ that is finitely generated over $\mathbb{Q}(\mathbb{U})$ containing the generators $\exp(\vec{a})$.

**Lemma 1.7.** Let $L$ be a finitely generated extension of $\mathbb{Q}(\mathbb{U})$ containing $\exp(\vec{a})$. Then there are $c_1, \ldots, c_t \in K$ linearly independent over $\sqrt{-1}\pi$ such that for every $\lambda_1, \ldots, \lambda_k \in L$, all the nondegenerate solutions of $[x^a]$ in $\Gamma^*$ are in $\mathbb{U} \cdot [\exp(\vec{c})]$.

Proof. ...
2. Specialisations and reduction to a number field

We first remark the following easy observation, whose proof is in Lemme 4 of \[8\], which we restate as Lemma 2.3 below (note that there, \( R \) is a finitely generated \( \mathbb{Q} \)-algebra, but the conclusion is stronger).

Lemma 2.1. Let \( R \) be a subring of \( \bar{\mathbb{Q}}[S] \), where \( S \) is a finite subset of \( \mathbb{C} \). Suppose that \( b_1, \ldots, b_q \) are elements of \( R \) and let \( q' \) be the linear dimension over \( \bar{\mathbb{Q}} \) of \( \tilde{b} \). Then there are ring homomorphisms \( \phi_1, \ldots, \phi_{q'} \) from \( R \) to \( \bar{\mathbb{Q}} \) fixing \( k := R \cap \mathbb{Q} \) such that for every \( \alpha_1, \ldots, \alpha_q \in k \) with \( \alpha_1 b_1 + \cdots + \alpha_q b_q \neq 0 \) there is some \( i \in \{1, \ldots, q'\} \) with \( \phi_i(\alpha_1 b_1 + \cdots + \alpha_q b_q) \neq 0 \).

Proof: ...

In order to reduce our setting to a number field in the next section, we need to carefully choose a specialization to \( \bar{\mathbb{Q}} \). This is ensured by the density of closed points in specific subsets of the spectrum of any finitely generated \( \mathbb{Q} \)-algebra \( R \). Given such \( R \) and a polynomial \( Q \) over \( R \) irreducible in \( \text{Frac}(R)[X] \), denote by \( \Omega(Q) \) the collection of prime ideals \( p \) of \( R \) such \( Q \mod p \) has the same degree as \( Q \) and it is irreducible as a polynomial over \( \text{Frac}(R/p) \). Recall that a Hilbert set \( \Omega \) is a subset of \( \text{Spec}(R) \) which contains a finite intersection of non-empty open sets and sets of the form \( \Omega(Q) \).

Fact 2.2. Let \( R \) be a finitely generated \( \mathbb{Q} \)-algebra.

(i) Given a finitely generated subgroup \( G \) of \( R^\times \), there is a Hilbert set \( \Omega \) such that the residue map \( G \to (R/p)^\times \) is injective for every \( p \) in \( \Omega \).

(ii) For any Hilbert set \( \Omega \) in \( R \), the collection of maximal ideals contained in \( \Omega \) is dense in \( \text{Spec}(R) \).

Combining the above with the proof of Lemma 2.1, one obtains the following result.

Lemma 2.3. (Lemme 4 in \[8\])

Let \( R \) be a finitely generated \( \mathbb{Q} \)-algebra with largest subfield \( k \) and \( G \) a finitely generated subgroup of \( R^\times \). Suppose also that \( b_1, \ldots, b_q \) are elements of \( R \) that generate a \( \mathbb{Q} \)-linear space of dimension \( q' \). Then there are ring homomorphisms \( \phi_1, \ldots, \phi_{q'} \) from \( R \) into \( \bar{\mathbb{Q}} \) such that each \( \phi_i \) is injective on \( G \) and that for every \( \alpha_1, \ldots, \alpha_q \in k \) with \( \alpha_1 b_1 + \cdots + \alpha_q b_q \neq 0 \), there is \( i \in \{1, \ldots, q'\} \) with

\[ \phi_i(\alpha_1 b_1 + \cdots + \alpha_q b_q) \neq 0. \]

In order to bound the degrees of the roots of unity appearing in Lemma 2.1, we will need the following result.

Theorem 2.4. (Theorem 1 in \[11\])

Let \( F \) be a number field, \( a_0, a_1, \ldots, a_k \) in \( F \) and \( \zeta \) a root of unity of order \( Q \) such that

\[ a_0 + \sum_{j=1}^k a_j \zeta^{n_j} = 0 \]

with \( \gcd(Q, n_1, \ldots, n_k) = 1 \). Let \( \delta = |F \cap \mathbb{Q}(\zeta) : \mathbb{Q}| \) and suppose that for any nonempty proper subset \( I \) of \( \{0, 1, \ldots, k\} \) the sum \( \sum_{j \in I} a_j \zeta^{n_j} \neq 0 \). Then for each prime \( p \) and \( n > 0 \), if \( p^{n+1} \mid Q \), then \( p^n | 2\delta \) and

\[ k \geq \dim_F(F + F \zeta^{n_1} + \cdots + F \zeta^{n_k}) \geq 1 + \sum_{p|Q,p|Q\mid Q} \left[ \frac{p - 1}{\gcd(p, p - 1)} - 1 \right]. \]
In particular, the order $Q$ of $\zeta$ is bounded by a constant depending on $k$ and $\delta$ (and therefore $[F : Q]$).

The last result of this section concerns work from [7]. Work inside a number field $F$. For $t, r$ in $\mathbb{N}$ consider polynomials $Q_1, \ldots, Q_r$ over $F$ in $t$ many variables as well as a finite set $Z := \{a_{ji} : j = 1, \ldots, r; i = 1, \ldots, t\}$ in $F^\times$. We are interested in describing the set of tuples $\vec{m}$ in $\mathbb{Z}^t$ such that

\begin{equation}
\sum_{j=1}^{r} Q_j(\vec{m}) \prod_{i=1}^{t} a_{ji}^{m_i} = 0.
\end{equation}

For such an equation (**), let $H$ be the subgroup of those $\vec{m}$ in $\mathbb{Z}^t$ such that

\begin{equation}
\prod_{i=1}^{t} a_{ji}^{m_i} = \prod_{i=1}^{t} a_{j'i}^{m_i},
\end{equation}

for every $j, j' \in \{1, \ldots, r\}$.

Théorème 6 of [7] describes precisely the solutions of (**), however for our purposes the following simplified version suffices.

**Theorem 2.5.** Suppose that $H$ is trivial. Then there are constants $\delta, \eta$ depending only on $Z$ and the field $F$ such that if $\vec{m}$ in $\mathbb{Z}^t$ satisfies (**) and for every nonempty proper $J \subseteq \{1, \ldots, r\}$ the sum $\sum_{j \in J} Q_j(\vec{m}) \prod_{i=1}^{t} a_{ji}^{m_i}$ is nonzero, then

\[ ||\vec{m}|| \leq \delta \log ||\vec{m}|| + \eta, \]

where $||\vec{m}|| := \max_i |m_i|$.

**Remark.** The independence of the constants $\delta, \eta$ from the coefficients of $Q_i$ follows from the proof of [7]. Therefore, there is some $N \in \mathbb{N}$ such that if $\vec{m}$ satisfies a non-trivial equation (**), then $||\vec{m}|| \leq N$.

3. **The Main Theorem**

We have now all the necessary tools to prove the following result.

**Theorem 3.1.** (Assuming Schanuel’s conjecture) For an irreducible complex polynomial $p$ in two variables where both variables appear, the entire function $f(z) := p(z, \exp(z))$ has infinitely many algebraically independent zeros.

**Proof.** Here we keep the notations from the previous sections. In particular, $K$ is an algebraically closed subfield of $\mathbb{C}$ of finite transcendence degree containing $\pi$ and the coefficients of $p$.

Using Hadamard Factorization Theorem (see for instance [6]) and a result proved independently by Henson and Rubel [5] and by van den Dries [1], we have that $f(z) = p(z, \exp(z))$ has infinitely many zeros in $\mathbb{C}$ (for a proof of this, see [9]). Therefore in order to prove our theorem, it suffices to show that $f(z)$ has finitely many zeros in $K$.

Write

\[ p(X, Y) = \sum_{j=0}^{m} p_j(X)Y^j, \]
where \( p_j(X) \in K[X] \). Also set \( I = \{ j \in \{0, \ldots, m\} \mid p_j \neq 0 \} \). Since \( p \) is irreducible, 0 lies in \( I \). The set \( \{ z \in \mathbb{C} \mid p_j(z) = 0 \text{ for some } j \in I \} \) is finite. Hence in order to show that there are finitely many solutions in \( K \) to \( p(z, \exp(z)) = 0 \) we need only prove that

\[
W = \{ z \in K \mid \text{for some } j \in I \}
\]

is finite.

**Claim.** There are \( c_1, \ldots, c_{\ell'} \) in \( K \) linearly independent over \( \sqrt{-1}\pi \) such that

\[
W \subseteq \mathbb{Q}\pi \sqrt{-1} + \mathbb{Z}c_1 + \cdots + \mathbb{Z}c_{\ell'}.
\]

**Proof.** ...

\( \square \)

We now apply Theorem 2.4 to get a finer description of \( W \).

**Claim.** There is \( N \in \mathbb{N} \) such that if \( z \in W \) then there are \( k, m_1, \ldots, m_{\ell'} \) in \( \mathbb{Z} \) and \( 0 < n < N \) such that

\[
z = \frac{k2\pi \sqrt{-1}}{n} + \sum_{j=1}^{\ell'} m_j c_j
\]

**Proof.** ...

\( \square \)

**Remark.** Using this claim we may assume, after modifying \( f \) (finitely many times) that its zeroes in \( K \) are of the form

\[
l2\pi \sqrt{-1} + \sum_{j=1}^{\ell'} m_j c_j
\]

with \( l, m_1, \ldots, m_{\ell'} \in \mathbb{Z} \).

We have reduced the theorem to prove that there are only finitely many \((l, \vec{m})\) such that

\[
(***) \sum_{j \in I} p_j(l2\pi \sqrt{-1} + \sum_{j=1}^{\ell'} m_j c_j)(\vec{d}^\vec{m})^j = 0 \text{ and } p_j(l2\pi \sqrt{-1} + \vec{m} \cdot \vec{c}) \neq 0 \text{ for } j \in I.
\]

Let \( R \) be the \( \mathbb{Q} \)-algebra generated by the coefficients of \( p, \pi \sqrt{-1}, \vec{c}, \vec{d} \) and their inverses. Let \( G \) be the multiplicative subgroup of \( R^\times \) generated by \( \vec{d} \). Choose \( \phi_1, \ldots, \phi_q \) ring homomorphisms from \( R \) to \( \mathbb{Q} \) injective on \( G \) as in Lemma 2.3 and let \( F \) be the compositum field of all their images.

Let \((l, \vec{m})\) satisfy \((***)\) and choose \( \nu \) in \( \{1, \ldots, q\} \) such that

\[
\phi_\nu(p_0(l2\pi \sqrt{-1} + \sum_{j=1}^{\ell'} m_j c_j)) \neq 0.
\]

The map \( \phi_\nu \) transforms \((***)\) into

\[
\sum_{j \in I} p_{\nu j}(\vec{m}) \prod_{i=1}^{\ell'} (\phi_\nu(d_i^j))^{m_i} = 0,
\]
where \( p_{jl}(\vec{X}) \) is a polynomial in \((1 + t')\)-variables such that
\[
p_{jl}(\vec{m}) = \phi_{\nu}(p_{j}(l2\pi \sqrt{-1} + \vec{m} \cdot \vec{c})).
\]
We may assume that no subsum is zero. Hence applying Theorem 2.5 and the remark after it, there is \( T \) in \( \mathbb{N} \) independent of \( l \) such that \( ||\vec{m}|| \leq T \). The proof finishes by noting that for each \( \vec{m} \), there are finitely many \( l \)'s satisfying (***). \( \square \)

References


